When do Lipschitz-free spaces have the (polynomial) Daugavet property?

SHELDON DANTAS

FACULTAD DE CIENCIAS
DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSIDAD DE GRANADA

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This talk is based on a work in progress with

⋆ Miguel **Martín**
Granada University, Spain

⋆ Yoël **Perreau**
Tartu University, Estonia

What do we do here?

Contents

- $\star$ In what world will we be working?

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- ⋆ In what world will we be working?
- ⋆ What property are we considering?

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⋆ What results do we have?

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- ⋆ What kind of problems do we want to tackle?
- ⋆ What results do we have?
- ⋆ What will we be doing now?

In what world will we be working?

Polynomials and Lipschitz-free spaces

Let $k \geq 0$ be given. Let X, Y be normed spaces.

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[2] 0-homogeneous polynomials are the constant maps

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- $\star$ $\mathcal{P}(^k X, Y) =$ k-homogeneous polynomials from X into Y[1][2].
- $\star$ $\mathcal{P}(X, Y) =$ all (continuous) polynomials of the form

$$P = \sum_{k=0}^{m} P_k$$

where $P_k \in \mathcal{P}(^k X, Y)$ for every $k = 0, 1, \dots, m$.

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- $\mathcal{P}(X, Y)$ has a structure of a normed space:

$$\|P\| := \sup_{x \in B_X} \|P(x)\| \quad (P \in \mathcal{P}(X, Y)).$$

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[2] 0-homogeneous polynomials are the constant maps

Polynomials and Lipschitz-free spaces

(Personal) relevant references about this topic:

- S. **Dineen**, Complex Analysis on infinite dimensional spaces
- J. **Mujica**, Complex analysis in Banach spaces
- P. **Hájek** and M. **Johanis**, Smooth Analysis in Banach spaces

Polynomials and Lipschitz-free spaces

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which is continuous and linear, that is, $\delta_x \in \mathrm{Lip}_0(M)^*, \forall x \in M$.

Polynomials and Lipschitz-free spaces

We then define the **Lipschitz-free space** over M by

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(a) $\mathrm{Lip}_0(M, X) = \mathcal{L}(\mathcal{F}(M), X)$ and, in particular,

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for $x \neq y$. It turns out that

$$B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\mathrm{Mol}(M)).$$

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Geodesic metric spaces

A geodesic metric space is a metric space which admits a geodesic[a] between any two of its points.

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A length metric space is a metric space such that the distance between any two points x and y is the infimum of lengths of paths between them.

- ⋆ Unlike in a geodesic metric space, the infimum does not have to be attained in a length space.
- ⋆ An example of a length space which is not geodesic is the Euclidean plane minus the origin: the points $(1,0)$ and $(-1,0)$ can be joined by paths of length arbitrarily close to 2, but not by a path of length 2.

Polynomials and Lipschitz-free spaces

Relevant references about this topic:

- N. **Weaver**, Lipschitz Algebras
- G. **Godefroy**, A survey on Lipschitz-free Banach spaces
- G. **Godefroy** and N. J. **Kalton**, Lipschitz-free Banach spaces
- R.J. **Aliaga**, E. **Pernecká** and R.J. **Smith**, De Leeuw representations of functionals on Lipschitz spaces,

What property are we considering?

The (polynomial) Daugavet property

Daugavet property

X has the DP if the norm equality

$$\| \mathrm{Id} + T \| = 1 + \| T \|$$

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- ⋆ $C(K)$ with K perfect [**Daugavet**, '63]
- ⋆ $L_1(\mu)$ with μ atomless [**Lozanovskii**, '66]
- ⋆ Some Banach algebras of holomorphic functions on an arbitrary Banach space [**Wojtaszczyk**, '92], [**Werner**, '97], [**Jung**, '23]

The (polynomial) Daugavet property

V. **Kadets**, M. **Martín**, A. **Rueda Zoca**, D. **Werner**
Banach spaces with the Daugavet property, preprint 2025

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☆ If $X \in \mathrm{DP}$, then X has a subspace isomorphic to ℓ_1.

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☆ If $X^* \in$ DP, then $X \in$ DP.

☆ $C[0,1]^* \notin$ DP.

Still surprising results

The (polynomial) Daugavet property

Still surprising results

V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, TAMS, 2000

Let X be a Banach space. TFAE:

(a) $X \in \mathrm{DP}$.

(b) $\forall x \in S_X, \forall \varepsilon > 0$, $\forall$ slice S of B_X, $\exists\ y \in S$ such that

$$\|x + y\| > 2 - \varepsilon.$$

(c) $\forall x \in S_X, \forall \varepsilon > 0$, $\forall$ slice S of B_X, $\exists\ y \in S$ such that

$$\|x - y\| > 2 - \varepsilon.$$

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- ⋆ $\mathrm{Lip}_0(M)$ has the DP whenever M is a length metric space. [Y. Ivakhno, V. Kadets, D. Werner, Math. Scand., 2007]

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- ⋆ $\mathrm{Lip}_0(M)$ has the DP whenever M is a length metric space.
  [Y. Ivakhno, V. Kadets, D. Werner, Math. Scand., 2007]
- ⋆ If M is compact, then $\mathrm{Lip}_0(M) \in$ DP iff $\mathcal{F}(M) \in$ DP.
  [Y. Ivakhno, V. Kadets, D. Werner, Math. Scand., 2007]

The (polynomial) Daugavet property

L. García-Lirola, T. Procházka, A. Rueda Zoca, JMAA, 2018

Let M be a complete pointed metric space. TFAE:

(a) M is a length metric space.

(b) $\mathrm{Lip}_0(M)$ has the Daugavet property.

(c) $\mathcal{F}(M)$ has the Daugavet property.

The polynomial Daugavet property

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Choi, **García**, **Maestre**, **Martín**, 2007, Studia Math.

The polynomial Daugavet property

Y.S. Choi, D. García, M. Maestre, M. Martín, 2007, Studia Math.

Let X be a Banach space. TFAE:

(a) X has the polynomial DP.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall P \in \mathcal{P}(X, \mathbb{K})$ with $\|P\| = 1$, $\exists y \in B_X$ and $\exists \omega \in \mathbb{T}$ such that

$$\operatorname{Re} \omega P(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

What kind of problems do we want to tackle?

Problem

DP $\Leftrightarrow$ *polynomial DP?*

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These properties are the same in

- $\star$ some $C(K)$-spaces
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  [M. Martín, A. Rueda Zoca]
- ⋆ JB*-triple (in particular C^*-algebras)
  [D. Cabezas, M. Martín, A. Peralta]

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- ⋆ Or not? Will they coincide also in $\mathcal{F}(M)$?
- ⋆ Expectation? As **Mingu** would say *물론이지! It is expected they are equivalent in this case, too!* Well, let's see...

What results do we have?

Our results

Strategy:

⋆ Analyze why $L_1[0,1]$ satisfies the polynomial DP.

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- $\star$ Analyze why $L_1[0, 1]$ satisfies the polynomial DP.
- $\star$ Generalize this proof to a broader context.
- $\star$ (Try to) adapt this new approach to Lipschitz-free spaces.

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First observations:

- ⋆ It turns out that there two main ingredients in the proof:

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First observations:

- ⋆ It turns out that there two main ingredients in the proof:
  - A "weakly sequential" property which yields big distances
  - and the Dunford-Pettis on $L_1[0,1]$.

Our results

Sequential Daugavet property

We say that X has the **sequential Daugavet property** if there exists a dense subset W of B_X such that for every $x \in S_X$ and for every $y \in W$, we can find a sequence $(y_n) \subseteq B_X$ such that

$$y_n \xrightarrow{w} y \qquad \text{and} \qquad \|x + y_n\| \to 2.$$

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Example (Essentially in [Daugavet-book, Lemma 4.5.9])

The space $L_1[0,1]$ has the sequential Daugavet property.

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Example (Essentially in [Daugavet-book, Lemma 4.5.9])

The space $L_1[0,1]$ has the sequential Daugavet property.

$\star$ Why to study that? As a **warm up** for Lipschitz-free spaces!

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Dunford-Pettis property

X has the **DPP** if every continuous weakly compact operator $T : X \to Y$ transforms weakly compact sets in X into norm-compact sets in Y.

- $\star$ In other words, X has DPP if, for every Y, every weakly compact operator $T : X \to Y$ maps weakly convergent sequences to norm converge sequences.

Our results

Proposition [Daugavet-book, Proposition 4.5.4]

If $X = c_0$ or $X = L_1(\mu)$, then every polynomial $P \in \mathcal{P}(X, \mathbb{K})$ is weakly sequentially continuous, that is,

$$x_n \xrightarrow{w} x \Rightarrow P(x_n) \to P(x).$$

More generally, the same holds true whenever X has the DPP.

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Remark

The proof that $L_1[0,1]$ has the polynomial DP finishes then from this together with the fact that it satisfies the sequential DP.

Our results

Recall that we are dealing here with Lipschitz-free spaces!

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Recall that we are dealing here with Lipschitz-free spaces!
We then wonder

when does $\mathcal{F}(M)$ have the Dunford-Pettis property?

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- ⋆ In particular, $\mathcal{F}(\ell_2)$ **fails** the DPP.
- ⋆ It is **not known** whether $\mathcal{F}(\mathbb{R}^n)$ has the DPP or not. (Antonín **Procházka** private communication)

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As **Geunsu** would say *There is no way, man!*

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What does it mean? That the Dunford-Pettis property in a smaller subspace would do the job!

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- $\star$ Then, we reproduce the $L_1[0,1]$ proof on each one of them.
- $\star$ Finally, we check that all of them are still far away from all the elements of the Lipschitz-free space.

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Our results: M geodesic $\Rightarrow \mathcal{F}(M)$ polynomial DP

Proof: While proving the sequential DP for $\mathcal{F}(M)$, we obtain, for every $\nu \in W$, a sequence (ν_n) which lives in the free space

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These geodesics are δ-separated. So, it is isomorphic to

$$\left(\bigoplus_{i=1}^{k} \mathcal{F}\left(\mathrm{Im}(\gamma_i)\right)\right) \oplus \ell_1^{k-1}$$

and then to an ℓ_1-sum of k copies of $L_1[0,1]$ and ℓ_1^{k-1}.
A. **Godard**, Tree metrics and their Lipschitz-free spaces

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A. **Godard**, Tree metrics and their Lipschitz-free spaces

Thus, to $L_1[0,1]$ itself (by the Pełczyński decomposition method). So, it has the DPP. This finishes the proof. ■

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Theorem (S.D., Y. Perreau, M. Martín, 2025+)

Let M be complete metric space. TFAE:

(1) M is length.

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(4) $\mathcal{F}(M)$ has the **strong diameter 2 property**.

(5) $\mathcal{F}(M)$ has the **diameter 2 property**.

(6) $\mathcal{F}(M)$ has the **slice diameter 2 property**.

(7) $B_{\mathcal{F}(M)}$ does not have **strongly exposed points**.

(8) $\mathrm{Lip}_0(M)$ has the **Daugavet property**.

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Let X be a Banach space and $x \in S_X$. We say that

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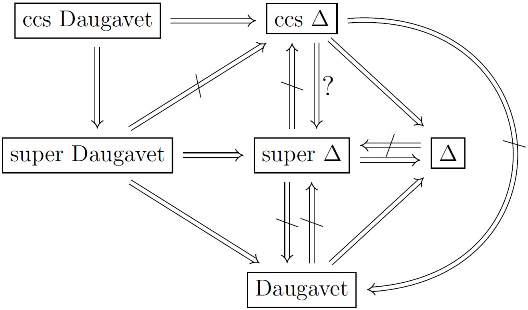
Delta- and Daugavet points in Banach spaces
T. A. **Abrahamsen**, R. **Haller**, V. **Lima**, and K. **Pirk**

Diametral notions for elements of the unit ball of a Banach space
M. **Martín**, Y. **Perreau** and A. **Rueda Zoca**

Our results



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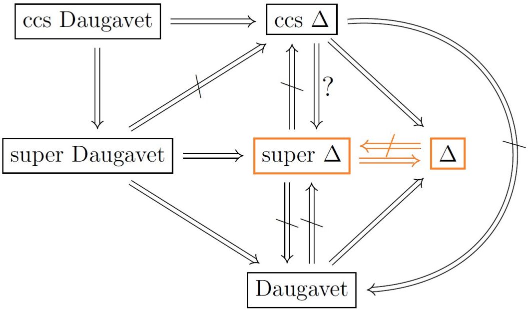

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Proof: We use the "support splitting lemma" once again. ■

It is worth mentioning that the "support splitting lemma" was already used (with a different proof) in

- The Lipschitz-free over a length space is LASQ but never ASQ
  R. **Haller**, J.K. **Kaasik**, and A. **Ostrak**
- Weakly almost square Lipschitz-free spaces
  J.K. **Kaasik** and T. **Veeorg**

WHAT WILL WE BE DOING NOW?

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  - ⋆ Actually, there are more...

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Weak operator Daugavet property (M. Martín and A. Rueda Zoca)

X has the WODP if, given $x_1, \ldots, x_n \in S_X$, $\varepsilon > 0$, a slice S of B_X and $x' \in B_X$, we can find $x \in S$ and $T : X \to X$ with

$$\|T\| \leq 1 + \varepsilon, \quad \|T(x) - x'\| < \varepsilon \quad \text{and} \quad \|T(x_i) - x_i\| < \varepsilon$$

for every $i = 1, \ldots, n$.

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- $\star$ $X \in$ WODP $\Rightarrow \hat{\bigotimes}_{\pi,s,N} X \in$ WODP $\Rightarrow \hat{\bigotimes}_{\pi,s,N} X \in$ DP.
- $\star$ L_1-preduals with the DP $\in$ WODP and $L_1(\mu, Y)$, μ atomless, $\forall Y \in$ WODP.

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- $\star$ M geodesic $\Leftrightarrow$ $\mathcal{F}(M)$ has the sequential DP?
- $\star$ X has the sequential DP $\Rightarrow$ X contains ℓ_2 or c_0?

Thank you very much for your attention!