The Daugavet property is equivalent to the polynomial Daugavet property

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This talk is based on a joint work (started in the BIRS-IMAG conference *Functional and Metric Analysis and their Interactions*) with

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Tartu University, Estonia

What do we do here?

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- ⋆ In what world will we be working?
- ⋆ What property are we considering?

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- ⋆ In what world will we be working?
- ⋆ What property are we considering?
- ⋆ What kind of problems do we want to tackle?
- ⋆ What results do we have?

In what world will we be working?

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Let X, Y be Banach spaces over $\mathbb{K}$ and $N \in \mathbb{N}$. A mapping P from X into Y is called an N-**homogeneous polynomial** if we can find an N-linear symmetric map $F : X^N \to Y$ (meaning that

$$F(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = F(x_1, \dots, x_N)$$

holds true for every permutation σ of $\{1, \dots, N\}$ and for every N-tuple $(x_1, \dots, x_N) \in X^N$) such that

$$P(x) = F(x, \dots, x)$$

for every $x \in X$.

(Homogeneous) Polynomials

- ⋆ $\mathcal{P}(^N X, Y) =$ N-homogeneous polynomials from X into Y[1][2].

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- In $\mathcal{P}(X, Y)$, we define

$$\|P\| := \sup_{x \in B_X} \|P(x)\| \quad (P \in \mathcal{P}(X, Y)).$$

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- $\star$ A polynomial P is a **rank-one** if $P(x) = p(x)y_0$ for every $x \in X$, where $p \in \mathcal{P}(X)$ and $y_0 \in Y$.

[3] if its closure is weakly compact

(Homogeneous) Polynomials

(Personal) relevant references about this topic:

- S. **Dineen**, Complex Analysis on infinite dimensional spaces
- J. **Mujica**, Complex analysis in Banach spaces
- P. **Hájek** and M. **Johanis**, Smooth Analysis in Banach spaces

WHAT PROPERTY ARE WE CONSIDERING?

The Daugavet property

Daugavet property (DPr, for short)

X has the DPr if the norm equality

$$\| \mathrm{Id} + T \| = 1 + \| T \|$$

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- ⋆ $C(K)$ with K perfect[4] [**Daugavet**, '63]
- ⋆ $L_1(\mu)$ with μ atomless [**Lozanovskii**, '66]
- ⋆ Some Banach algebras of holomorphic functions on Banach spaces [**Wojtaszczyk**, '92], [**Werner**, '97], [**Jung**, '23]

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Recent reference

V. **Kadets**, M. **Martín**, A. **Rueda Zoca**, D. **Werner**
Banach spaces with the Daugavet property
Preprint
2025

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☆ $C[0,1]^* \notin$ DPr.

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V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, TAMS, 2000

X has the DPr if and only if B_X is equal to the closed convex hull of the set

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for every $x \in B_X$ and $\varepsilon > 0$. In other words, for every $x \in S_X$ and every slice[a] S of B_X, we have

$$\sup_{y \in S} \|x + y\| = 2.$$

[a] For $x^* \in X^*$ and $\varepsilon > 0$, a **slice of a set** A is a set of the form

$$S(A, x^*, \varepsilon) = \{x \in A : x^*(x) > x^*(A) - \varepsilon\}.$$

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Shvidkoy's lemma

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Let X be a Banach space with the DPr. Then, for every $x \in S_X$ and $\varepsilon > 0$, the set

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$\star$ A lot of results on the DPr can be proved by using this lemma.

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Choi, **García**, **Maestre**, **Martín**, 2007, Studia Math.

The polynomial Daugavet property

Y.S. Choi, D. García, M. Maestre, M. Martín, 2007, Studia Math.

Let X be a Banach space. TFAE:

(a) X has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall P \in \mathcal{P}(X, \mathbb{K})$ with $\|P\| = 1$, $\exists y \in B_X$ and $\exists \omega \in \mathbb{T}$ (a modulus-one scalar) such that

$$\operatorname{Re} \omega P(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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  [Y.S. **Choi**, D. **García**, M. **Maestre**, M. **Martín**, 2014]
- $\star$ Spaces of Lipschitz functions and Lipschitz-free spaces
  [L. **García-Lirola**, T. **Procházka**, A. **Rueda Zoca**, 2018]
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- ⋆ L_1-preduals
  [M. **Martín** and A. **Rueda Zoca**, 2022]
- ⋆ C^*-algebras and JB*-triples
  [E.R. **Santos**, 2014]
  [D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

What kind of problems do we want to tackle?

Problem

Is it true that *DPr* $\iff$ *polynomial DPr?*

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(1) In view of the geometric characterization of the polynomial DPr, by Shvidkoy's characterization, we have the polynomial DPr from the classical DPr provided that the polynomials we are working with are **weakly continuous** on B_X.

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[D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

(2) However, it is known that in infinite dimensional Banach spaces, the only weakly continuous homogeneous polynomials are those of **finite type**, that is, of the form

$$P(x) = \sum_{j=1}^{m} \alpha_j \varphi_j^N(x)$$

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[R. **Aron** and J.B. **Prolla**, 1980]

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[D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

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[J. **Ferrera**, J. **Gomez Gil** and J.L. **González Llavona**, 1983]

Problem

Conclusion

From these results in

(1) [D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

(2) [R. **Aron** and J.B. **Prolla**, 1980]

(3) [J. **Ferrera**, J. **Gomez Gil** and J.L. **González Llavona**, 1983]

we **cannot** expect to get the equivalence between the DPr and the polynomial DPr as a simple consequence of the characterization due to Shvidkoy's.

Problem

The general strategy used so far

[5] If every continuous weakly compact operator $T : X \to Y$ transforms weakly compact sets in X into norm-compact sets in Y.

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- ⋆ If X has the **Dunford-Pettis property**[5], then all polynomials on X are weakly sequentially continuous. [**Dineen**, Complex Analysis on Infinite-Dimensional Spaces]

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- ⋆ In the previous results:
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  - ⋆ Use the weak sequential continuity of polynomials (or their Aron-Berner extensions) for theses sequences.

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- ⋆ One exception: [**Cabezas, Martín, Peralta**, '24]
  - ⋆ A completely different topology (the strong* topology).
  - ⋆ To show some sequential continuity for polynomials under this specific topology.

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  - and the **Dunford-Pettis** on $L_1[0, 1]$.

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- ⋆ In particular, $\mathcal{F}(\ell_2)$ **fails** the DPP.
- ⋆ It is **not known** whether $\mathcal{F}(\mathbb{R}^n)$ has the DPP or not. (Antonín **Procházka** private communication)

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(4) Some consequences.

What results do we have?

The tool(s)

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The **polynomial-star topology** of X^{**} is the smallest topology on X^{**} for which a net (x_α) in X^{**} converges to a point x in X^{**} if and only if $\hat{p}(x_\alpha) \longrightarrow \hat{p}(x)$ for every scalar-valued polynomial p on X, where $\hat{p}$ denotes the Aron-Berner extension[a] of p to X^{**}.

[a] [R.M. Aron and P.D. Berner, 1978]

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- ⋆ We will use the ideas of the proof of this result.

Our results

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Recall that TFAE:

(a) X has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall P \in \mathcal{P}(X, \mathbb{K})$ with $\|P\| = 1$, $\exists y \in B_X$ and $\exists \omega \in \mathbb{T}$ (a modulus-one scalar) such that

$$\operatorname{Re} \omega P(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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$$\left\| x + \sum_{i=1}^{N} y_i \right\| > N + 1 - \frac{\varepsilon}{N+1} \tag{1}$$

and

$$|F(y_{i_1}, \ldots, y_{i_m}) - F(y, \ldots, y)| < \varepsilon \tag{2}$$

for every m-linear form $F \in \mathcal{F}$ and for every $i_1 < \cdots < i_m \in \{1, \ldots, N\}$.

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Claim: We will construct $y_1, \ldots, y_N$ such that, for every $n \in \{1, \ldots, N\}$,

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$$|F(y_{i_1}, \ldots, y_{i_{k-1}}, y_{i_k}, y, \ldots, y) - F(y_{i_1}, \ldots, y_{i_{k-1}}, y, y, \ldots, y)| < \frac{\varepsilon}{N}$$

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Proof of the Claim: Use Shvidkoy's lemma twice!

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$$\left| F(y_{i_1}, \ldots, y_{i_m}) - F(y, \ldots, y) \right| < \cdots < m \cdot \frac{\varepsilon}{N} \leq \varepsilon$$

as we are done. $\square$

Thank you very much for your attention!