Main Result: Indeed, on one hand, it is proved in [DG89] that taking N large enough and letting

$$y_{\mathcal{F}} := \frac{1}{N}\sum_{i=1}^{N} y_i,$$

we can deduce from (2) that

$$|F(y_{\mathcal{F}},\dots,y_{\mathcal{F}}) - F(y,\dots,y)| < 2\varepsilon$$

$\forall$ $F\in\mathcal{F}$. On the other hand, by H-B and arguing by contradiction, from (1) we can get a functional $f\in S_{X^*}$ such that

$$\operatorname{Re} f(x) > 1-\frac{\varepsilon}{N+1} \quad \text{and} \quad \operatorname{Re} f(y_i) > 1-\frac{\varepsilon}{N+1},\ \forall i=1,\dots,N$$

Using this functional,

$$\|x+y_{\mathcal{F}}\| \geq \operatorname{Re} f(x) + \frac{1}{N}\sum_{i=1}^{N}\operatorname{Re} f(y_i) > 2-\varepsilon.$$

Now we choose as direct set the product of $(0,1)$ by the set of all finite families of continuous symmetric multilinear forms on X to construct the desired net.

$\boxed{n=1}$ Apply Shmidkov's lemma to find a net $(y_\alpha) \subseteq B_X$ which converges weakly to y and such that $\|x+y_\alpha\| \longrightarrow 2$. Since the map $z \mapsto F(z, y, \dots, y)$ is linear and continuous (hence also weak continuous) for every $F \in \mathcal{F}$, we have

$$F(y_\alpha, y, \dots, y) \longrightarrow F(y, y, \dots, y).$$

Therefore, we can find α such that $\|x+y_\alpha\| > 2-\varepsilon$ and such that

$$|F(y_\alpha, y, \dots, y) - F(y, y, \dots, y)| < \frac{\varepsilon}{N},$$

$\forall F \in \mathcal{F}$. So, $y_1 := y_\alpha$ satisfies the above requirements.

Assume that we've constructed $y_1, \dots, y_n \in B_X$ satisfying the above conditions for $n \in \{1, \dots, N-1\}$.

Apply Shvidkoy's lemma to the point

$$\frac{x + \sum_{i=1}^{n} y_i}{\left\| x + \sum_{i=1}^{n} y_i \right\|} \in S_X$$

to obtain a net $(y_\alpha) \subseteq B_X$ which converges weakly to y and such that

$$\left\| \frac{x + \sum_{i=1}^{n} y_i}{\left\| x + \sum_{i=1}^{n} y_i \right\|} + y_\alpha \right\| \longrightarrow 2.$$

As before, we have that given any m-linear $F \in \mathcal{F}$, $k \leq m-1$ and $i_1 < \dots < i_k \leq n$, the map

$$z \longmapsto F(y_{i_1}, \dots, y_{i_k}, z, y, \dots, y)$$

is linear and continuous, so

$$F(y_{i_1}, \dots, y_{i_k}, y_\alpha, y, \dots, y) \longrightarrow F(y_{i_1}, \dots, y_{i_k}, z, y, \dots, y).$$

Since there is only a finite number of linear forms and finitely many possible choices for the multi-indices $i_1 < \dots < i_k$ for each of these, we can find α such that

$$|F(y_{i_1}, \dots, y_{i_k}, y_\alpha, y, \dots, y) - F(y_{i_1}, \dots, y_{i_k}, y, y, \dots, y)| < \frac{\varepsilon}{N}$$

for all of such a choice, and additionally such that

$$\left\| \frac{x + \sum_{i=1}^{n} y_i}{\|x + \sum_{i=1}^{n} y_i\|} + y_\alpha \right\| > 2 - \frac{\varepsilon}{\|x + \sum_{i=1}^{n} y_i\|}.$$

By using the induction hypothesis and applying the Remark# for $\lambda = \frac{1}{\|x + \sum_{i=1}^{n} y_i\|} \in (0,1)$

Remark: Let X be a BS, $\varepsilon \in (0,1)$ and $x, y \in B_X$ be such that $\|x+y\| > 2-\varepsilon$. Then, $\forall \lambda \in (0,1)$, we have

$$\|x + \lambda y\| > 1 + \lambda - \varepsilon.$$

we get that

$$\left\| x + \sum_{i=1}^{n} y_i + y_\alpha \right\| > 1 + \left\| x + \sum_{i=1}^{n} y_i \right\| - \varepsilon$$

$$> n + 2 - (n+1)\varepsilon .$$

So, $y_{n+1} := y_\alpha$ and we're done.

<u>Corollary:</u> ~~If~~ $X \in \mathcal{D}Pr$, then $X \in$ (polynomial $\mathcal{D}Pr$).

<u>PROOF:</u> Let $x \in S_X$, $p \in \mathcal{P}(X)$ with $\|p\| = 1$ and $\varepsilon > 0$. Pick $y \in B_X$ and $w \in \mathbb{K}$, $|w| = 1$, to be such that $\mathrm{Re}\, w p(y) > 1 - \varepsilon$.

Apply the previous theorem to the points $\overline{w} x \in S_X$ and $y \in B_X$, and find a net $(y_\alpha) \subseteq B_X$ which converges to y in the weak polynomial topology and such that $\|x + w y_\alpha\| \longrightarrow 2$.

Since in particular $p(y_\alpha) \longrightarrow p(y)$, we can find α such that $\mathrm{Re}\, w p(y_\alpha) > 1 - \varepsilon$ and still $\|x + w y_\alpha\| > 2 - \varepsilon$. By the geometric characterization of the polynomial $\mathcal{D}Pr$, we're done. ■