Revisiting Shvidkoy's characterization of the DPr via the polynomial weak topology

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The 21st ILJU School of Mathematics
Banach Spaces and Related Topics

January 26th, 2026, Seoul, South Korea

This talk is based on a recent joint work with

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- ⋆ What property are we considering?
- ⋆ What kind of problems do we want to tackle?

What do we do here?

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- ⋆ What property are we considering?
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What do we do here?

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In what world will we be working?

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Let X, Y be Banach spaces over $\mathbb{K}$ and $N \in \mathbb{N}$. A mapping P from X into Y is called an N-**homogeneous polynomial** if we can find an N-linear symmetric map $F : X^N \to Y$ (meaning that

$$F(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) = F(x_1, \ldots, x_N)$$

holds true for every permutation σ of $\{1, \ldots, N\}$ and for every N-tuple $(x_1, \ldots, x_N) \in X^N$) such that

$$P(x) = F(x, \ldots, x)$$

for every $x \in X$.

(Homogeneous) Polynomials

⋆ $\mathcal{P}(^N X, Y) =$ N-homogeneous polynomials from X into Y[1][2].

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- $\star$ $\mathcal{P}(^N X, Y) =$ N-homogeneous polynomials from X into Y[1][2].
- $\star$ $\mathcal{P}(X, Y) =$ all (continuous) polynomials of the form

$$P = \sum_{k=0}^{m} P_k$$

where $P_k \in \mathcal{P}(^k X, Y)$ for every $k = 0, 1, \ldots, m$.

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- ⋆ $\mathcal{P}(X) =$ all scalar-valued continuous polynomials on X.

- In $\mathcal{P}(X, Y)$, we define

$$\|P\| := \sup_{x \in B_X} \|P(x)\| \quad (P \in \mathcal{P}(X, Y)).$$

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- ⋆ A polynomial P is a **rank-one** if $P(x) = p(x)y_0$ for every $x \in X$, where $p \in \mathcal{P}(X)$ and $y_0 \in Y$.

[3] if its closure is weakly compact

(Homogeneous) Polynomials

(Personal) relevant references about this topic:

- S. **Dineen**, Complex Analysis on infinite dimensional spaces
- J. **Mujica**, Complex analysis in Banach spaces
- P. **Hájek** and M. **Johanis**, Smooth Analysis in Banach spaces

What property are we considering?

What property are we considering? No pause! You already know it!

The Daugavet property

Daugavet property (DPr, for short)

X has the DPr if the norm equality

$$\| \mathrm{Id} + T \| = 1 + \| T \|$$

holds for all rank-one bounded linear operators on X.
Equivalently, for all weakly compact linear operators[a] on X.

[a] $T : X \to Y$ is weakly compact if $\overline{T(B_X)}$ is a weakly compact set in Y

- ⋆ $C(K)$ with K perfect [**Daugavet**, '63]
- ⋆ $L_1(\mu)$ with μ atomless [**Lozanovskii**, '66]
- ⋆ Some Banach algebras of holomorphic functions on Banach spaces [**Wojtaszczyk**, '92], [**Werner**, '97], [**Martín** and **Oikhberg**, '04], [**Jung**, '23]

The Daugavet property

A SURPRISING RESULT (THE MOST SURPRISING PERHAPS?)

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V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, TAMS, 2000

X has the DPr if and only if B_X is equal to the closed convex hull of the set

$$\{y \in B_X : \|x + y\| > 2 - \varepsilon\}$$

for every $x \in B_X$ and $\varepsilon > 0$.

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for every $x \in B_X$ and $\varepsilon > 0$. In other words, for every $x \in S_X$ and every slice[a] S of B_X, we have

$$\sup_{y \in S} \|x + y\| = 2.$$

[a] For $x^* \in X^*$ and $\varepsilon > 0$, a **slice of a set** A is a set of the form

$$S(A, x^*, \varepsilon) = \{x \in A : x^*(x) > x^*(A) - \varepsilon\}.$$

The Daugavet property

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Shvidkoy's lemma

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Let X be a Banach space with the DPr. Then, for every $x \in S_X$ and $\varepsilon > 0$, the set

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$$y_\alpha \xrightarrow{w} y \qquad \text{and} \qquad \|x + y_\alpha\| \to 2.$$

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$\star$ A lot of results on the DPr can be proved by using this lemma.

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Choi, **García**, **Maestre**, **Martín**, 2007, Studia Math.

The polynomial Daugavet property

Y.S. Choi, D. García, M. Maestre, M. Martín, 2007, Studia Math.

Let X be a Banach space. TFAE:

(a) X has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall P \in \mathcal{P}(X, \mathbb{K})$ with $\|P\| = 1$, $\exists y \in B_X$ and $\exists \omega \in \mathbb{T}$ (a modulus-one scalar) such that

$$\operatorname{Re} \omega P(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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- ⋆ C^*-algebras and JB*-triples
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What kind of problems do we want to tackle?

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You know that one, too!

Main problem

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(2) However, it is known that in infinite dimensional Banach spaces, the only weakly continuous homogeneous polynomials are those of **finite type**, that is, of the form

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[J. **Ferrera**, J. **Gomez Gil** and J.L. **González Llavona**, 1983]

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Conclusion

From these results in

(1) [D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

(2) [R. **Aron** and J.B. **Prolla**, 1980]

(3) [J. **Ferrera**, J. **Gomez Gil** and J.L. **González Llavona**, 1983]

we **cannot** expect to get the equivalence between the DPr and the polynomial DPr as a simple consequence of the characterization due to Shvidkoy's.

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The general strategy used so far

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[**Dineen**, Complex Analysis on Infinite-Dimensional Spaces]

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- In the previous results:
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  - Use the weak sequential continuity of polynomials (or their Aron-Berner extensions) for theses sequences.

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- One exception: [**Cabezas, Martín, Peralta**, '24]
  - A completely different topology (the strong* topology).
  - To show some sequential continuity for polynomials under this specific topology.

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First observations:

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  - A "weakly sequential" property which yields big distances
  - and the **Dunford-Pettis property** on $L_1[0,1]$.

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- $\star$ In particular, $\mathcal{F}(\ell_2)$ **fails** the DPP.

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- $\star$ If X is a separable infinite dimensional reflexive space, then $\mathcal{F}(X)$ cannot have the DPP. Indeed, in this case X is complemented in $\mathcal{F}(X)$ and it is known that a Banach space with DPP cannot have complemented reflexive spaces.
- $\star$ In particular, $\mathcal{F}(\ell_2)$ **fails** the DPP.
- $\star$ It is **not known** whether $\mathcal{F}(\mathbb{R}^n)$ has the DPP or not. (A. **Procházka** - private communication)

It's time to change the approach!

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(3) Then, we will have DPr $\Leftrightarrow$ polynomial DPr.

(4) Some consequences.

WHAT RESULTS DO WE HAVE?

What results do we have?

You might guess already otherwise it would be very disappointing!

The tool(s)

[T.K. Carne, B. Cole and T.W. Gamelin, TAMS, 1989]

The **weak polynomial topology** on a Banach space X

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The **polynomial-star topology** of X^{**} is the smallest topology on X^{**} for which a net (x_α) in X^{**} converges to a point x in X^{**} if and only if $\hat{p}(x_\alpha) \longrightarrow \hat{p}(x)$ for every scalar-valued polynomial p on X, where $\hat{p}$ denotes the Aron-Berner extension[a] of p to X^{**}.

[a] [R.M. Aron and P.D. Berner, 1978]

The tool(s)

[A.M. Davie and T.W. Gamelin, Theorem 2, 1989]

$B_{X^{**}}$ is equal to the polynomial-star closure of B_X in X^{**}.

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$B_{X^{**}}$ is equal to the polynomial-star closure of B_X in X^{**}.

- ⋆ This provides a polynomial-star Goldstine theorem.
- ⋆ We will use the ideas of the proof of this result.

Our results

[D., Martín, Perreau, 2025]
(Shvidkoy's lemma for the weak polynomial topology)

Let X be a Banach space with the DPr.

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Let X be a Banach space with the DPr. Then, for every $x \in S_X$ and $y \in B_X$, we can find a net $(y_\alpha) \subseteq B_X$ which converges to y in the weak polynomial topology of B_X and such that $\|x + y_\alpha\| \to 2$.

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Corollary (DPr $\Leftrightarrow$ polynomial DPr)

A Banach space X with the DPr satisfies the polynomial DPr.

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Corollary (DPr $\Leftrightarrow$ polynomial DPr)

A Banach space X with the DPr satisfies the polynomial DPr.

Recall that TFAE:

(a) X has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall p \in \mathcal{P}(X, \mathbb{K})$ with $\|p\| = 1$, $\exists y \in B_X$ and $\exists \omega \in S_{\mathbb{K}}$ such that

$$\operatorname{Re} \omega p(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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- $\star$ Apply the theorem to the points $\overline{\omega} x \in S_X$ and $y \in B_X$.

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- ⋆ Apply the theorem to the points $\overline{\omega} x \in S_X$ and $y \in B_X$.
- ⋆ By doing this, we can find $(y_\alpha) \subset B_X$ converging to y in the weak polynomial topology and such that $\|x + \omega y\| \to 2$.
- ⋆ In particular, $p(y_\alpha) \longrightarrow p(y)$ and so we can find α such that

$$\operatorname{Re} \omega p(y_\alpha) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y_\alpha\| > 2 - \varepsilon$$

and this exactly what we wanted. ■

Our results

All we need to do now is to prove the main result.

[D., Martín, Perreau, 2025]
(Shvidkoy's lemma for the weak polynomial topology)

Let X be a Banach space with the DPr. Then, for every $x \in S_X$ and $y \in B_X$, we can find a net $(y_\alpha) \subseteq B_X$ which converges to y in the weak polynomial topology of B_X and such that $\|x + y_\alpha\| \to 2$.

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$$\left\| x + \sum_{i=1}^{N} y_i \right\| > N + 1 - \frac{\varepsilon}{N+1} \tag{1}$$

and

$$|F(y_{i_1}, \ldots, y_{i_m}) - F(y, \ldots, y)| < \varepsilon \tag{2}$$

for every m-linear form $F \in \mathcal{F}$ and for every $i_1 < \cdots < i_m \in \{1, \ldots, N\}$.

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for every m-linear form $F \in \mathcal{F}$ and for every $i_1 < \cdots < i_m \in \{1, \ldots, N\}$. This yields the desired net by choosing as direct set the product of $(0,1)$ by the set of all finite families of continuous symmetric multilinear forms on X.

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Claim (use original Shvidkoy's lemma twice!): We construct $y_1, \ldots, y_N$ such that, for every $n \in \{1, \ldots, N\}$,

$$\left\| x + \sum_{i=1}^{N} y_i \right\| > n + 1 - n\xi \quad \text{and}$$

$$|F(y_{i_1}, \ldots, y_{i_{k-1}}, y_{i_k}, y, \ldots, y) - F(y_{i_1}, \ldots, y_{i_{k-1}}, y, y, \ldots, y)| < \frac{\varepsilon}{N}$$

for every m-linear symmetric form $F \in \mathcal{F}$, every $k \leq m$ and every $i_1 < \cdots < i_k \leq n$.

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Proving the claim, we have that

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$$\begin{aligned}
\left| F(y_{i_1}, \ldots, y_{i_m}) - F(y, \ldots, y) \right| \leq & \left| F(y_{i_1}, \ldots, y_{i_m}) - F(y_{i_1}, \ldots, y_{i_{m-1}}, y) \right| \\
& + \left| F(y_{i_1}, \ldots, y_{i_{m-1}}, y) - F(y_{i_1}, \ldots, y_{i_{m-2}}, y, y) \right| \\
& + \cdots + \left| F(y_{i_1}, y, \ldots, y) - F(y, \ldots, y) \right| \\
& < m \cdot \frac{\varepsilon}{N} \leq \varepsilon.
\end{aligned}$$

and we are done. ■

Is there more to tell?

A consequence

⋆ This result shows in particular that, for **real** Banach spaces with the DPr, we have

$$\|p\| = \sup_{x \in S_X} |p(x)|$$

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- ⋆ This result is always true for **complex** spaces. (Maximum Modulus Principle)
- ⋆ **Real** polynomials can easily attain their norms in the interior of the ball. (the polynomial $x \mapsto 1 - \langle x, x \rangle$ on a Hilbert space)
- ⋆ For a systematic study we refer to [S.D. and R. Medina, 2024]

Daugavet centers

⋆ $G \in \mathcal{L}(X, Y)$ is a **Daugavet center** if $\|G + T\| = \|G\| + \|T\|$ holds for every rank-one $T \in \mathcal{L}(X, Y)$.
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- ⋆ $Q \in \mathcal{P}(X, Y)$ is **polynomial Daugavet center** if $\|Q + P\| = \|Q\| + \|P\|$ for every rank-one polynomial $P \in \mathcal{P}(X, Y)$.
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- ⋆ X has the DPr if and only if Id is a Daugavet center.
- ⋆ **(Shvidkoy-type lemma for Daugavet centers)** Let X be a Banach space and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$ be a Daugavet center. Then for every $x \in B_X$ and $y \in S_Y$, there exists a net (x_α) in B_X such that $x_\alpha \to x$ weakly and $\|y + G(x_\alpha)\| \to 2$.
  [T.V. Bosenko, 2010]

Daugavet centers

[D., Martín, Perreau, 2025]

Let X be a Banach space and $G \in \mathcal{L}(X, Y)$ be a Daugavet center. For every $x \in B_X$ and $y \in S_Y$, there exists a net (x_α) in B_X such that $x_\alpha \to x$ in the weak polynomial topology and $\|y + G(x_\alpha)\| \to 2$.

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Every (linear) Daugavet center is a polynomial Daugavet center.

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Corollary

Every (linear) Daugavet center is a polynomial Daugavet center.

$\star$ The linearity of G above is crucial: we **do not know** if, given a polynomial $Q \in \mathcal{P}(X, Y)$ such that $\|Q + T\| = \|Q\| + \|T\|$ for every rank-one $T \in \mathcal{L}(X, Y)$, we actually have that the same equality holds for T being a rank-one polynomial from X into Y.

Can we use this approach somewhere else?

CAN WE USE THIS APPROACH SOMEWHERE ELSE?

⋆ *How to construct in general (homogeneous) polynomials from linear operators?* ⋆

Thank you very much for your attention!