Def: Let K be a compact Hausdorff space, X be a (real) Banach space and $T: \mathcal{C}(K) \to X$ be a bdd linear operator.

(a) A point $s_0 \in K$ is called a point of diffusion for T if $\forall \varepsilon > 0$, $\exists$ a ngh $\mathcal{U}(s_0)$ st

$$g \in \mathcal{C}(K),\ \|g\|_\infty \leq 1,\ g(s) = 0,\ \forall s \notin \mathcal{U}(s_0)$$
$$\Rightarrow \|Tg\| < \varepsilon.$$

(b) T is called almost diffuse if the set of pts of diffusion is dense in K.

Result: If $T: \mathcal{C}(K) \to \mathcal{C}(K)$ is almost diffuse, then

$$\|Id + T\| = 1 + \|T\|.$$

Proof: Let $\varepsilon > 0$ and assume that $\varepsilon < \|T\|$. Pick a function $f \in \mathcal{C}(K)$, $\|f\|_\infty = 1$, such that

$$\|T(f)\|_\infty > \|T\| - \varepsilon,$$

that is,

$$|(Tf)(t_0)| > \|T\| - \varepsilon$$

for some $t_0 \in K$.

Therefore, the inequality

$$|(Tf)(t)| > \|T\| - \varepsilon$$

holds true in some ngh $V(t_0)$. Since T is almost diffuse, $V(t_0)$ contains a point of diffusion, say s_0. By def there is a ngh $\mathcal{U}(s_0) \subseteq V(t_0)$ of s_0 st

$$g \in \mathcal{E}(K), \|g\|_\infty \leq 1, g(s) = 0, \forall s \notin \mathcal{U}(s_0) \Rightarrow \|Tg\| < \varepsilon.$$

Let $s \in \mathcal{U}(s_0)$. Pick some $h \in \mathcal{E}(K)$ such that

$$\|h\|_\infty \leq 1, \quad h = f \text{ on } K \setminus \mathcal{U}(s_0) \quad \text{and}$$

$$h(s) := \frac{(Tf)(s)}{|(Tf)(s)|}$$

(since $s \in V(t_0)$, the denominator here is nonzero).

It follows that

- $\|h - f\|_\infty \leq 2$, • $h - f = 0$ on $K \setminus \mathcal{U}(s_0)$

and, therefore, $\|T(h-f)\| < 2\varepsilon$.

Consequently,

$$
\begin{aligned}
\|Id + T\| &\geq \|h + Th\|_\infty \\
&= \|h + T(-f+h) + T(f)\|_\infty \\
&\geq \|h + Tf\|_\infty - 2\varepsilon \\
&\geq |h(s) + (Tf)(s)| - 2\varepsilon \\
&= \left| \frac{(Tf)(s)}{|(Tf)(s)|} + (Tf)(s) \right| - 2\varepsilon \\
&= \big|\, |(Tf)(s)| + 1 \,\big| - 2\varepsilon \\
&= 1 + |(Tf)(s)| - 2\varepsilon \\
&> 1 + \|T\| - \varepsilon - 2\varepsilon \\
&= 1 + \|T\| - 3\varepsilon .
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, $\|Id + T\| = 1 + \|T\|$. ∎