1   Lecture #1: Tuesday, February 17th, 2026

1.1   Introduction

In this course, we will study Ordinary Differential Equations (ODEs, for short) together with Numerical Analysis. These are two broad and important areas of research in Mathematics and Applied Mathematics, and each of them could easily justify a separate course. Nevertheless, we will consider them together, and there are good reasons for doing so. Throughout the semester, we will aim to convince the reader of this choice.

These two subjects share several important features, one of which is their strong usefulness and applicability in real-life situations. The question of how they can be useful in practice is addressed by Mathematical Modeling. How does this work? We are interested in solving real-world problems by describing them mathematically, typically through *equations*. In doing so, we construct a new, virtual reality in which mathematics becomes a tool for providing *solutions* to these problems. After that, one needs to go back to life and see what these solutions can tell us about real life.

In most of these problems, we are interested in understanding how a given situation changes over time. A simple way to illustrate this idea is to note that when we move from one place to another, the temperature of our body changes accordingly, and it also evolves with time. This means that, when modeling real-life situations, one should naturally expect derivatives to appear. In this way, we are led to equations that involve derivatives and therefore to the study of differential equations.

It turns out that we have already encountered this type of equation in Calculus. For instance, we know how to solve the equation $y'(x) = 2x$*. Indeed, by integrating y', we obtain $y(x) = x^2 + C$ for any constant $C \in \mathbb{R}$. However, if we modify the equation slightly, the problem changes completely. For example, consider the equation $y' = 2x + y$. Previously, we solved the problem using integration, but this approach no longer works here, and we must adopt a different method to find a solution. In this case, there exists an analytical procedure, that is, a mathematical method that provides an explicit solution. If we modify the equation once more and consider $y' = 2x + y^2$, the situation changes again. In this case, there is no general procedure that yields an explicit solution. This shows that differential equations are extremely sensitive, and that even a small modification can significantly increase their level of difficulty.

The question now is what to do when we are faced with a differential equation that cannot be solved explicitly. In such cases, we turn to numerical methods, which allow us to approximate the solution of the given problem. For instance, the exact solution of the equation $x^5 = 13$ is $x = \sqrt[5]{13}$. If we slightly modify this problem to obtain $x^5 + x = 13$, then there is no algebraic procedure that allows us to find an explicit solution for this new equation. Notice, however, that this equation does have a solution, since the continuous function $f(x) = x^5 + x - 13$ has derivative $f'(x) = x^4 + 1$, which is always positive, and satisfies $\lim_{x \to \pm\infty} f(x) = \pm\infty$. Therefore, by Bolzano's Theorem[†], the equation has at least one root. By using Numerical Analysis, we can apply suitable methods to approximate this solution as accurately as desired, even though it cannot be found explicitly.

This is precisely what we will do in this course. We will learn analytical methods for obtaining explicit solutions of (differential) equations, and when this is not possible, we will develop numerical methods to

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*In this course, we will not write this equation as $y'(x) = 2x$, but simply as $y' = 2x$. Before solving this problem, the reader should understand that y is a function of x, that is, $y = y(x)$

[†]If a continuous function defined on an interval takes both positive and negative values, then it must vanish at some point.

approximate their solutions.

These notes are **entirely** based on Prof. Petr Habala's classes that can be found on

https://math.fel.cvut.cz/en/people/habala/teaching/den-e.html

and also on his Youtube videos about this course on

https://www.youtube.com/playlist?list=PLQL6z4JeTTQnIbkbPliqX1QZiXs35RQjq.

1.2 ODEs of order 1

We begin the course by studying some of the most relevant and widely used methods for solving ordinary differential equations (ODEs, for short) analytically. Our approach is to present each method together with plenty of examples in order to ensure a thorough understanding. Keep in mind that we will also have problem sessions, in which we will put into practice most of the topics covered in the theory classes throughout the semester.

An ordinary differential equation (ODE, for short) is an equation that involves derivatives of different orders of a function y. By a solution of an ODE, we mean a function that satisfies this equation. More precisely, we have the following definition.

Definition 1.1. By an explicit **ordinary different equation of order** n we mean any equation of the form
$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$
where f is a function of n variables. By a **solution on an open interval** I we mean any function $y = y(x)$ that has all derivatives up to order n on I and for all $x \in I$ satisfies
$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)).$$

Notice that the unknown in an ODE is a function. Notice also that the name "ordinary" means that the derivatives we are dealing with are derivative of function with one variable, so we are not using partial derivatives but only ordinary ones.

Example 1.2. The ODE
$$y'''(x) = y(x) + \frac{y'(x)}{y'(x) + x} + e^x$$
is an ODE of order 3. Notice that the variable x can be interpreted as time. In fact, in many real-life situations, the variable x represents time, and we will denote it by t instead. However, x may also represent position or another physical quantity, depending on the context. This will become clear in each specific problem we consider. As mentioned before, we will often omit the explicit dependence on the variable x in our notation. Thus, the previous ODE can be written as
$$y''' = y + \frac{y'}{y' + x} + e^x.$$

Let us consider another example.

Example 1.3. Consider an object falling from the air to the ground under the action of gravity, as illustrated in Figure 1. For simplicity, assume that the object moves only vertically. The vertical axis represents the height (or elevation) of the object above the ground, denoted by $y(t)$, while the horizontal axis represents time t. If we analyze the motion at a given instant, we can describe the forces acting on the object mathematically. Since the second derivative of the position corresponds to the acceleration, and gravity is the only force acting on the object, we obtain the following ordinary differential equation:
$$y''(t) = -g. \tag{1}$$

Here, g denotes the acceleration due to gravity. The negative sign appears because the motion is directed downward, which corresponds to the negative direction of the vertical axis.
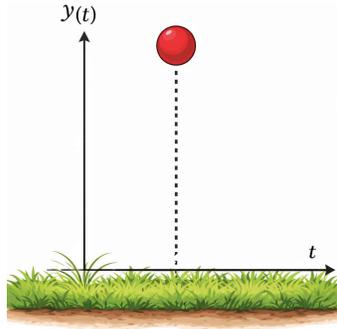
Figure 1: An object falling from the air

How do we solve this ODE? We integrate both sides. This yields

$$y'(t) = -gt + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Integrating once again, we obtain

$$y(t) = -\frac{1}{2}gt^2 + Ct + D, \tag{2}$$

where $D \in \mathbb{R}$ is another arbitrary constant.

Notice that substituting (2) into (1) shows that (2) satisfies the differential equation for every $t \in \mathbb{R}$. Therefore, we may write the general solution as

$$y(t) = -\frac{1}{2}gt^2 + Ct + D \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

Let us note that there are *infinitely many* solutions (corresponding to the infinitely many choices of the constants C and D in (3)) to the ODE appearing in Example 1.3. This is reasonable, since in our physical model the motion of the object under the influence of gravity depends on the initial conditions, in particular on the height from which the object is released (see Figure 2).
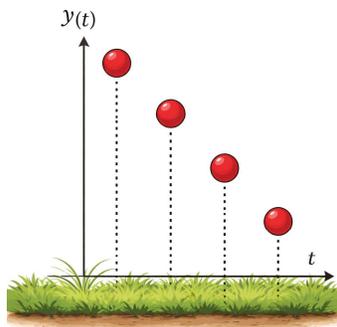

Figure 2: Different outcomes for different initial conditions

In order to obtain a unique solution, we need two initial conditions, since (3) depends on two parameters. For example, if we prescribe the values $y(t_1) = y_1$ and $y(t_2) = y_2$, then the problem admits a unique solution of (1). However, in most physical situations we are given the initial position and the initial velocity of the object. That is, if at time t_0 the height is y_0 and the initial velocity is v_0, then the initial conditions are given by $y(t_0) = y_0$ and $y'(t_0) = v_0$. Let us see an example.

Example 1.4. Let us consider the ODE $y'' = -g$ as before, with initial conditions $y(0) = y_0$ and $y'(0) = v_0$. Taking into account the general solution (3), we substitute $y(0) = y_0$ and obtain

$$y_0 = -\frac{1}{2}g \cdot 0^2 + C \cdot 0 + D \Rightarrow D = y_0.$$

Since $y' = -gt + C$, we also have

$$v_0 = -g \cdot 0 + C \Rightarrow C = v_0.$$

Therefore, the unique solution of (1) is given by

$$y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0, \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

Let us observe that in our example does not make sense to write $t \in \mathbb{R}$ in (4) as in our case t stands for time. So, we can rewrite it by putting $t \geqslant 0$ instead.

Let us introduce some conventions and clarify the terminology. If the set of all solutions of a given ODE on an interval I can be expressed by a single formula involving parameters, we call this formula the **general solution** of the ODE. Any specific solution obtained by fixing the values of these parameters is called a **particular solution**.

Definition 1.5. Consider an ODE of order n,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}).$$

By an **initial value problem** (IVP, for short), or a **Cauchy problem** for this equation, we mean a problem of the form

(1) ODE: $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$;

(2) initial conditions: $y(x_0) = y_0, ; y'(x_0) = y_1, ; \ldots, ; y^{(n-1)}(x_0) = y_{n-1}$.

Here, $x_0, y_0, y_1, \ldots, y_{n-1}$ are fixed real numbers.

Let us note that, in general, solving an ODE is far from easy. In fact, we may ask how to solve the one appearing in Example 1.2 above. The presence of initial conditions does not help much if we do not know how to solve the ODE itself, that is, if we are unable to determine its general solution. In the next section, we will focus on a very special class of ODEs for which an effective method of solution can be developed.

1.3 Solving ODEs using separation

In this section, we will present a method for very particular ODEs called separation. In order to do so, we start with an example.

Example 1.6. Consider the ODE

$$y' = 2xe^{-y}.$$

How can we solve it? Notice that we cannot simply integrate both sides with respect to x, since y depends on x (that is, $y = y(x)$). Let us rewrite the equation using Leibniz notation, $y' = \frac{dy}{dx}$:

$$\frac{dy}{dx} = 2xe^{-y}.$$

Now we separate the variables by bringing the terms involving y to the left-hand side and the terms involving x to the right-hand side, and then we integrate both sides:

$$\int e^y dy = \int 2x dx.$$

Since these integrals are elementary, we obtain

$$e^y = x^2 + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. To obtain the general solution, we solve for y by applying ln to both sides:

$$y = \ln(x^2 + C), \quad \text{for every } x \in \mathbb{R} \text{ such that } x^2 + C > 0. \tag{5}$$

In particular, if $C \geq 0$, then $x^2 + C > 0$ for all $x \in \mathbb{R}$, and the solution is defined on all of $\mathbb{R}$. If $C < 0$, then the condition $x^2 + C > 0$ is equivalent to $x^2 > -C$, that is, $|x| > \sqrt{-C}$, so the solution is defined on

$$(-\infty, -\sqrt{-C}) \cup (\sqrt{-C}, +\infty).$$

To determine which domain (and which value of C) is relevant in a concrete situation, we need initial conditions (see Definition 1.5). This means that the general solution is given by (5).

Let notice that not always we can separate an ODE at the same way we have done in Example 1.6 above. For instance, the ODE

$$\frac{dy}{dx} = x + y$$

cannot be separated. In fact, we have the following definition.

Definition 1.7. By a **separable ODE** we mean any ODE that can be expressed in the form

$$y' = g(x)h(y)$$

for some functions $g = g(x)$ and $h = h(y)$.

Besides the fact that not every ODE is separable, let us notice that, following the ideas from Example 1.6, we may run into difficulties when trying to compute the integrals that arise after separating the variables x and y. For instance, we do not know how to evaluate the integral

$$\int e^{-x^2} dx$$

using the standard techniques covered in a typical Calculus course. There is a third issue we may encounter: even after integrating, we may not be able to isolate y explicitly. For instance, if after integration we obtain $y^{13} + y = x + C$, there is no algebraic way to solve for y and write the solution explicitly.

Example 1.8. Consider the ODE

$$y' = 2xe^{-y}$$

with initial condition $y(0) = 1$. Observe that we solved this ODE in Example 1.6, where its general solution was found to be

$$y(x) = \ln(x^2 + C), \ x^2 + C > 0.$$

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Using the initial condition $y(0) = 1$, we obtain

$$\ln(0^2 + C) = 1 \Rightarrow C = e.$$

Therefore, the solution of the corresponding Cauchy problem is

$$y_a(x) = \ln(x^2 + e), \ \ x \in \mathbb{R}.$$

Suppose now that the initial condition is $y(2) = 0$. Using again the general solution, we obtain

$$\ln(2^2 + C) = 0 \Rightarrow \ln(4 + C) = 0 \Rightarrow 4 + C = 1 \Rightarrow C = -3.$$

Therefore, the solution of this Cauchy problem is given by $y_b(x) = \ln(x^2 - 3)$, with the restriction $x^2 - 3 > 0$. This means that x must belong either to $(-\infty, -\sqrt{3})$ or to $(\sqrt{3}, +\infty)$. Which interval should we choose? Since the initial condition is $y(2) = 0$, we must select the interval $(\sqrt{3}, +\infty)$ as it contains the point $x = 2$.

Let us now generalize what we have done before by considering a general separable ODE. Suppose that $y'(x) = g(x)h(y)$. Separating the variables, we obtain

$$\frac{y'(x)}{h(y(x))} = g(x).$$

Integrating both sides, we have

$$\int \frac{y'(x)}{h(y(x))}dx = \int g(x)dx.$$

If we set $w = y(x)$, then $dw = y'(x)dx$, and therefore

$$\int \frac{1}{h(w)}dw = \int g(x)dx.$$

Since $w = y$, this can be written as

$$\int \frac{1}{h(y)}dy = \int g(x)dx.$$

If $H(y)$ denotes an antiderivative of $1/h(y)$ and $G(x)$ denotes an antiderivative of $g(x)$, then

$$H(y) = G(x) + C,$$

where $C \in \mathbb{R}$ is a constant. If the function H is invertible, we can solve this equation for y and thus obtain an explicit solution of the ODE. With a bit more of precision, we have proved the following result.

Theorem 1.9. Consider a separable ODE $y' = g(x)h(y)$. Assume that g is continuous on some open integral I and h is continuous on some open interval J. If $h \neq 0$ on J, then there is a solution to the given equation on I. Let $G(x)$ be an antiderivative of $g(x)$ on I and $H(y)$ be antiderivative of $1/h(y)$ on J. If H has an inverse function H_{-1}, then a general solution of the given equation can be expressed as

$$y(x) = H_{-1}(G(x) + C).$$

Notice that, in order to prove Theorem 1.9, it suffices to mimic the arguments from Example 1.6. We encourage the student to learn the step-by-step procedure for solving separable ODEs, rather than memorizing the statement of the above theorem. Let us consider another example.

Example 1.10. Consider the following first-order ODE

$$y' = 2x(y - 1).$$

Notice that this equation is separable, since it can be written in the form $y' = g(x)h(y)$ with $g(x) = 2x$ and $h(y) = y - 1$. Rewriting the equation using Leibniz notation, we have

$$\frac{dy}{dx} = 2x(y - 1).$$

Separating the variables and integrating both sides, we obtain

$$\int \frac{dy}{y - 1} = \int 2x dx.$$

Notice that in here we have to assume that $y \neq 1$. Evaluating these integrals, we find

$$\ln|y - 1| = x^2 + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Exponentiating both sides yields

$$|y - 1| = e^{x^2 + C} = e^C e^{x^2} = D e^{x^2},$$

where $D \in \mathbb{R}$ with $D \neq 0$ is another arbitrary constant. Absorbing the absolute value into the constant, we can write

$$y - 1 = D e^{x^2},$$

and therefore

$$y = 1 + D e^{x^2}, \ x \in \mathbb{R}$$

is the general solution of the ODE.

What if $y = 1$? To reply to this question, let us consider the following fact.

Fact 1.11. Consider a separable ODE $y' = g(x)h(y)$. If y_0 satisfies $h(y_0) = 0$, then the constant function $y(x) = y_0$ is a solution to the given ODE on any open interval $I \subseteq D(g)$ so-called **stationary solution**.

Going back to Example 1.10, we have the $y_0 = 1$ is a stationary solution. Therefore, the solution for this example is

$$y(x) = 1 + D e^{x^2} \text{ for all } x \in \mathbb{R} \quad \text{or} \quad y(x) = 1 \text{ for all } x \in \mathbb{R}.$$

Notice that we can still put these two formulas together by allowing D to be 0. This implies that the general solution is given by $y(x) = 1 + D e^{x^2}$ for all $x \in \mathbb{R}$, where $D \in \mathbb{R}$ is arbitrary.

The algorithm to solve separable ODEs by separation is given as follows.

Algorithm 1.12 (Solving separable ODE by separation). Consider an ODE of the form $y' = g(x)h(y)$.

(1) Write the equation as $\frac{dy}{dx} = g(x)h(y)$. Move all x (including dx) to the right and all y (including dy) to the left, and add integration signs:

$$\int \frac{dy}{h(y)} = \int g(x) dx.$$

(2) Explore the possibility that $h(y) = 0$ leading to possible stationary solutions.

(3) Assuming that $h(y) \neq 0$, integrate both sides.

$$\int \frac{dy}{h(y)} = \int g(x)dx \Rightarrow H(y) = G(x) + C.$$

(4) If possible, express y as a function of x:

$$H(y) = G(x) + C \Rightarrow y(x) = H_{-1}(G(x) + C).$$

(5) Exploring the given equation and the solution we obtained, determine conditions for its validity.

(6) If an initial condition is given, determine the corresponding particular solution and also the maximal integral of its validity.