2 Lecture #2: Wednesday, February 18th, 2026

2.1 Two more examples using separation

Let us consider another example.

Example 2.1. Consider the following first-order ODE

$$y' = \frac{x}{y}.$$

Notice that $y \neq 0$. This implies that the solutions cannot cross the x-axis. Let us solve this equation using the method of separation of variables. Writing the equation in Leibniz notation, we have

$$\frac{dy}{dx} = \frac{x}{y} \quad \Rightarrow \quad \int y dy = \int x dx.$$

Integrating both sides, we obtain

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C.$$

Multiplying both sides by 2, this can be written as

$$y^2 = x^2 + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant (we rename $2C$ as C for simplicity). Therefore, the general solution is

$$y = \pm\sqrt{x^2 + C} \text{ with } \; x^2 + C > 0$$

keeping in mind that y must be nonzero.

We now consider four different initial conditions, which lead to four distinct problems.

(a) Suppose that $y(1) = 2$. From the general solution, we obtain

$$2 = \sqrt{1^2 + C} \Rightarrow 1 + C = 4 \Rightarrow C = 3.$$

Thus, the solution in this case is

$$y_a(x) = \sqrt{x^2 + 3} \text{ for all } x \in \mathbb{R}.$$

(b) Suppose that $y(5) = -3$. Using the general solution, we obtain

$$-3 = -\sqrt{5^2 + C} \Rightarrow 9 = 25 + C \Rightarrow C = -16.$$

Hence,

$$y_b(x) = -\sqrt{x^2 - 16} \text{ with } x^2 - 16 > 0.$$

Solving $x^2 - 16 > 0$, we find that $x \in (-\infty, -4)$ or $x \in (4, +\infty)$. Since the initial condition is given at $x = 5$, we choose the interval $(4, +\infty)$. Therefore,

$$y_b(x) = -\sqrt{x^2 - 16} \text{ for all } x \in (4, +\infty).$$

(c) Suppose that $y(-5) = 4$. Then

$$4 = \sqrt{(-5)^2 + C} \Rightarrow 16 = 25 + C \Rightarrow C = -9.$$

Thus,

$$y_c(x) = \sqrt{x^2 - 9} \text{ with } x^2 - 9 > 0.$$

This implies that $x \in (-\infty, -3)$ or $x \in (3, +\infty)$. Since the initial condition is given at $x = -5$, we select the interval $(-\infty, -3)$. Hence,

$$y_c(x) = \sqrt{x^2 - 9} \text{ for all } x \in (-\infty, -3).$$

(d) Suppose that $y(1) = 0$. As observed earlier, this situation cannot occur because y must always be different from zero. Therefore, in this case, the ODE admits no solution.

Example 2.2. Let us rewrite the equation from Example 2.1. We now consider the ODE

$$y'y = x.$$

Notice that the difference in this case is that there is no restriction preventing y from being zero. Therefore, we can follow the same arguments as before and conclude that the general solution of this ODE is

$$y(x) = \pm\sqrt{x^2 + C} \text{ with } x^2 + C \geqslant 0.$$

Moreover, if we consider the initial condition $y(1) = 0$, which previously did not admit any solution, we now obtain

$$0 = \pm\sqrt{1^2 + C} \Rightarrow 0 = 1 + C \Rightarrow C = -1.$$

Hence, the corresponding solutions are

$$y_d(x) = \pm\sqrt{x^2 - 1} \text{ for all } x \in [1, +\infty).$$

Notice that there is no way to decide whether the positive or negative branch should be chosen, since both satisfy the initial condition $y(1) = 0$. From a mathematical point of view, this is problematic, as we will later see that initial value problems are expected to admit unique solutions under suitable assumptions.

2.2 Applications of ODEs of order 1 - Exponential growth

In this section, we will study applications of first-order ordinary differential equations. By the end of the section, we hope to convince the reader of the usefulness of these equations in real-life situations.

Example 2.3 (Money)**.** Let us give a mathematical interpretation of the statement “the more money you have, the more money you make”. Let $y(t)$ denote the amount of money in a bank account at time t. Clearly, this quantity depends on time. The idea that the more money you have, the more money you make can be expressed by saying that the rate of change of y increases with y itself. In other words, the derivative of y is proportional to y. This leads to the following ODE

$$y' = ry,$$

where r is a constant representing the rate of growth. Notice that this is a separable ODE and also notice that $y = 0$ is a stationary solution. Let us separate the variables to obtain

$$\int \frac{dy}{y} = \int r dt.$$

Integrating both sides, we get

$$\ln |y| = rt + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Thus, $|y| = e^{rt+C}$, that is, $y = \pm e^C \cdot e^{rt}$. Hence, we obtain the general solution

$$y(t) = D \cdot e^{rt} \text{ for all } t \geq 0,$$

where $D \in \mathbb{R}$. Notice that this family of solutions also includes the stationary solution $y = 0$. Now, if $y(0) = y_0$, which represents the initial amount of money in the bank, then

$$y_0 = D \cdot e^{r \cdot 0} = D \Rightarrow D = y_0.$$

This implies that

$$y(t) = y_0 \cdot e^{rt} \text{ for all } t \geq 0$$

is the solution of this Cauchy problem.

The phenomenon from Example 2.3 is called **exponential growth**. Notice that exponential growth does not fully reflect reality, since not everyone's money grows exponentially. Nevertheless, it is our task as human beings to interpret whether a mathematical solution provides a reasonable model for a given real-life situation. There are, however, situations in which this type of model fits reality much better.

Example 2.4 (Radioactive decay)**.** One important example is radioactive decay. While in the money example the amount of money grows proportionally to its current value, in radioactive decay the quantity of a radioactive substance decreases proportionally to the amount that is still present. More precisely, let $y(t)$ denote the amount of a radioactive substance at time t. The physical principle behind radioactive decay states that the rate at which the substance decays is proportional to the amount of substance remaining. Mathematically, this leads to the same ODE as in Example 2.3, namely $y' = ry$, but now the constant r is *negative*. The negative sign reflects the fact that the quantity of the substance is decreasing over time rather than increasing. Solving this ODE as in the previous example, we obtain $y(t) = y_0 e^{rt}$, where y_0 is the initial amount of the substance. Since $r < 0$, the exponential term e^{rt} decreases as t increases, and therefore $y(t)$ tends to zero as time goes on. This behavior accurately models the observed decay of radioactive materials.

Example 2.5 (Rabbits)**.** Another situation in which exponential growth provides a reasonable model is the growth of a population of rabbits. Let $y(t)$ denote the number of rabbits at time t. If we assume that food and space are abundant and that the reproduction rate is proportional to the current population size, then the rate of change of y is proportional to y itself. This leads again to the ODE $y' = ry$, where $r > 0$ represents the growth rate of the population. Solving this equation as before, we obtain $y(t) = y_0 e^{rt}$, where y_0 is the initial number of rabbits. Since r is positive, the population grows exponentially over time. This model is reasonable for short periods of time, although in reality it eventually breaks down due to limited resources and environmental constraints.

2.3 Applications of ODEs of order 1 - Newton's law of cooling

Example 2.6 (Ice cream in the desert)**.** Another important application of first-order ODEs is Newton's law of cooling. Suppose we are in the desert, where the ambient temperature is $T_D = 50°$C. To cool down, we take an ice cream out of a fridge. As time passes, heat from the environment acts on the ice cream and its temperature increases. Let $y(t)$ denote the temperature of the ice cream at time t. Newton's law of cooling states that the rate of change of the temperature is proportional to the difference between the ambient temperature and the object's temperature. This leads to the ODE

$$y' = r(50 - y),$$

where $r > 0$ is a constant depending on the physical properties of the system. Notice that $y = 50$ is a stationary solution, corresponding to thermal equilibrium with the environment. If $y \neq 50$, we can solve the equation by separation of variables:

$$\int \frac{dy}{50 - y} = \int r dt.$$

Integrating, we obtain

$$-\ln |50 - y| = rt + C.$$

That is, $50 - y = \pm e^{-C} \cdot e^{-rt}$. Therefore, we have a family of solutions of the form

$$y(t) = 50 + De^{-rt}$$

valid for all $t \geq 0$, where $D \in \mathbb{R}$ depends on the initial temperature of the ice cream. Since $e^{-rt} \to 0$ as $t \to \infty$, the temperature $y(t)$ approaches 50°C over time. This means that, regardless of the initial temperature, the ice cream's temperature gradually converges to the ambient temperature of the desert. This behavior is called exponential relaxation and contrasts with exponential growth or decay, even though the mathematical structure of the solution is similar.

2.4 Existence of solutions: Peano and Picard theorems

Up to this point, we have focused on methods for explicitly solving ordinary differential equations. However, as we have already seen, many ODEs cannot be solved in closed form, even when initial conditions are given. This naturally leads to two fundamental questions:

(a) Does a solution to a given initial value problem exist?

(b) If it exists, is it unique?

The answers to these questions are provided by two fundamental results in the theory of ODEs, the **Peano existence theorem** and the **Picard–Lindelöf theorem**. The Peano theorem guarantees the existence of at least one solution under mild assumptions, while the Picard–Lindelöf theorem strengthens this result by ensuring both existence and uniqueness under stronger conditions. These theorems play a central role in understanding the behavior of differential equations beyond explicit solution techniques (see Figure 3).

Theorem 2.7 (Peano's theorem on existence)**.** Consider an ODE of the form

$$y' = f(x, y). \tag{6}$$

Let I, J be open intervals such that f is continuous on the set $I \times J$. Then, for all $(x_0, y_0) \in I \times J$, there is a solution of the IVP (6) with $y(x_0) = y_0$ on some neighborhood of x_0.

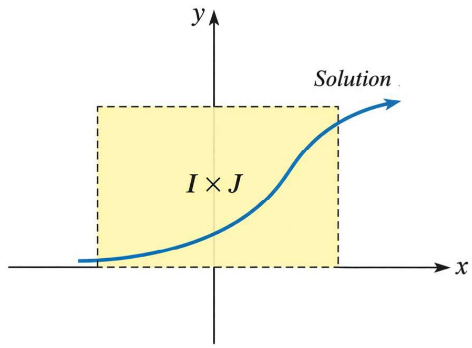

Figure 3: Peano's theorem on existence.

Notice that Peano's theorem does not say anything about the uniqueness of solutions. This limitation makes the result less useful for applications, since in most practical situations we are interested not only in the existence of a solution, but also in its uniqueness. For instance, let us consider once again the ODE $y' = 2x(y-1)$, which can be written in the standard form $y' = f(x, y)$ with $f(x, y) = 2x(y-1)$. The function $f(x, y)$ is a polynomial in the variables x and y, and therefore it is continuous on $\mathbb{R}^2$. As a consequence, for any initial condition $y(x_0) = y_0$, Peano's theorem guarantees the existence of at least one solution defined on some interval containing x_0. However, Peano's theorem does not provide any information about whether this solution is unique. In this particular example, we know from explicit computations that the solution is in fact unique, but this conclusion does not follow from Peano's theorem alone.

In order to guarantee not only the existence but also the uniqueness of solutions to an initial value problem, we need a condition stronger than mere continuity. This leads to the notion of Lipschitz functions, which plays a central role in the Picard–Lindelöf theorem.

Definition 2.8. Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be K**-Lipschitz** on I if there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leqslant K|x - y| \text{ for all } x, y \in I.$$

If such a constant exists, we say that f is Lipschitz on I.

Intuitively, the Lipschitz condition means that the function cannot oscillate or grow too fast: its slope is uniformly bounded. One important feature about Lipschitz functions we should take into account is the fact that every Lipschitz function is continuous.

Examples 2.9. Let us look at some examples.

(a) The function $f(x) = \sin x$ is Lipschitz on $\mathbb{R}$. Indeed, since $|f'(x)| = |\cos x| \leqslant 1$ for all x, the Mean Value Theorem[‡] implies that there is $c \in (x, y)$ such that

$$\frac{|\sin x - \sin y|}{|x - y|} \leqslant |\cos(c)| \leqslant 1.$$

[‡](Mean Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Thus, $\sin(x)$ is 1-Lipschitz.

(b) The function $f(x) = e^x$ is not Lipschitz on $\mathbb{R}$. Although it is continuous everywhere, its derivative $f'(x) = e^x$ is unbounded on $\mathbb{R}$. Hence, no global constant K can satisfy the Lipschitz condition. However, e^x is Lipschitz on any bounded interval.

(c) The function $f(x) = \sqrt{x}$, defined on $[0, +\infty)$, is continuous but not Lipschitz on intervals containing 0. Indeed, its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ blows up as $x \to 0^+$. As a consequence, no constant K can control the behavior of f near 0. This makes $\sqrt{x}$ a typical example of a function that is continuous but not Lipschitz.

Theorem 2.10 (Picard's theorem on existence and uniqueness)**.** Consider an ODE of the form

$$y' = f(x, y). \tag{7}$$

Let I, J be open intervals such that f is continuous on the $I \times J$ and there exists $K > 0$ such that for all $x \in I$, f is K-Lipschitz as a functions of y on J. Then, for all $(x_0, y_0) \in I \times J$ there exists a solution of the IVP (7) with $y(x_0) = y_0$ on some neighborhood of x_0 and this solution is unique on this neighborhood.

A useful consequence of Picard's theorem is the following. Notice that it avoids talking about Lipschitz functions.

Corollary 2.11. Consider an ODE of the form (7). If I, J are open intervals such that f is continuous and $\frac{\partial f}{\partial y}$ exists and it bounded on the set $I \times J$, then through every point $(x_0, y_0) \in I \times J$, there passes exactly one solution of the equation (7) and it can be extended to the boundary of $I \times J$.

Example 2.12. Consider again the ODE given by $y' = 2x(y - 1)$. Defining $f(x, y) = 2x(y - 1)$, we compute the partial derivative with respect to y:

$$\frac{\partial f}{\partial y} = 2x.$$

This derivative is bounded on any set of the form $(-M, M) \times \mathbb{R}$, with $M > 0$, since

$$\left| \frac{\partial f}{\partial y} \right| = |2x| \leqslant 2M.$$

Therefore, the function $f(x, y)$ is locally Lipschitz with respect to the variable y. As a consequence, we can apply Corollary 2.11, which guarantees the local existence and uniqueness of a solution for the given ODE, for any initial condition. In fact, we can say that through every point in $\mathbb{R}^2$ there passes a unique (local) solution.