

4 Lecture #3: Tuesday, February 24th, 2026

4.1 Analyzing solutions

Let us consider a Cauchy problem of the form

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases}$$

As we have seen previously, in some cases we are able to find an explicit family of solutions for an ODE. For instance, for certain equations we can obtain a picture such as the one shown in Figure 4, where several solution curves are displayed together. Let us notice that the commands appear on the top of the figure as well in case one wants to play with Maple.

```
> with(DEtools):  
plot([1, 1 + exp(x^2), 1 + exp(x^2)/2, 1 + exp(x^2)/5, 1 + exp(x^2)/15, 1 + exp(x^2)/55, 1 - exp(x^2), 1 - exp(x^2)/2, 1 - exp(x^2)/5, 1 - exp(x^2)/55], x=0..2, y=-1  
..3, color=navy);
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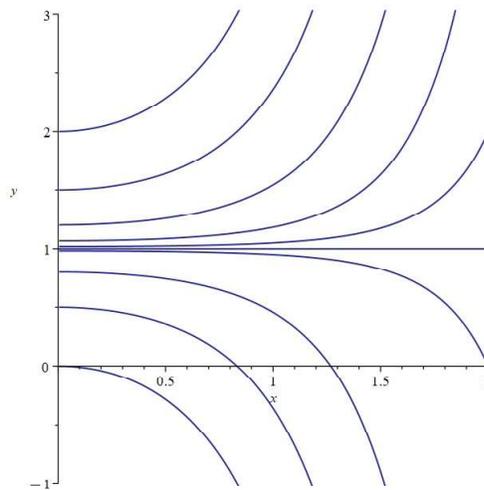


Figure 4: Peano's theorem on existence.

Obtaining such a picture is one of the main goals of this section. However, it is important to notice that not every ordinary differential equation can be solved analytically. For example, consider the equation

$$y' = \frac{y}{y-1}.$$

Although this equation may appear simple, there is no general procedure to find an explicit formula for its solutions. This raises the following question: how can we still obtain qualitative information about the solutions when an explicit solution is not available? To answer this question, we introduce the idea of analyzing solutions using slope fields. This method allows us to sketch the behavior of solutions without solving the equation explicitly.

4.2 Slope field

Example 4.1. Consider once again the ODE

$$y' = 2x(y - 1).$$

In this case, the solutions can be defined on the entire plane \mathbb{R}^2 as there is no restrictions for $f(x, y) = 2x(y - 1)$. Our goal is to determine where the solutions are increasing and where they are decreasing. Recall from the Calculus course that a function is increasing where its derivative is positive and decreasing where its derivative is negative. Here, the derivative is given by the ODE itself, that is, by $y' = 2x(y - 1)$. The derivative exists everywhere, so changes in monotonicity occur when

$$2x(y - 1) = 0.$$

This happens when $x = 0$ or $y = 1$. These two curves divide the plane into four regions (see Figure 5). Within each region, the sign of the derivative does not change. Let us analyze the sign of y' in each region (see Figure 5) - one can analyze it in particular points as $(1, 0)$, $(-1, 2)$, $(1, 2)$ and $(-1, 0)$:

- ★ If $x > 0$ and $y > 1$, then $y' > 0$, so solutions are increasing.
- ★ If $x > 0$ and $y < 1$, then $y' < 0$, so solutions are decreasing.
- ★ If $x < 0$ and $y > 1$, then $y' < 0$, so solutions are decreasing.
- ★ If $x < 0$ and $y < 1$, then $y' > 0$, so solutions are increasing.

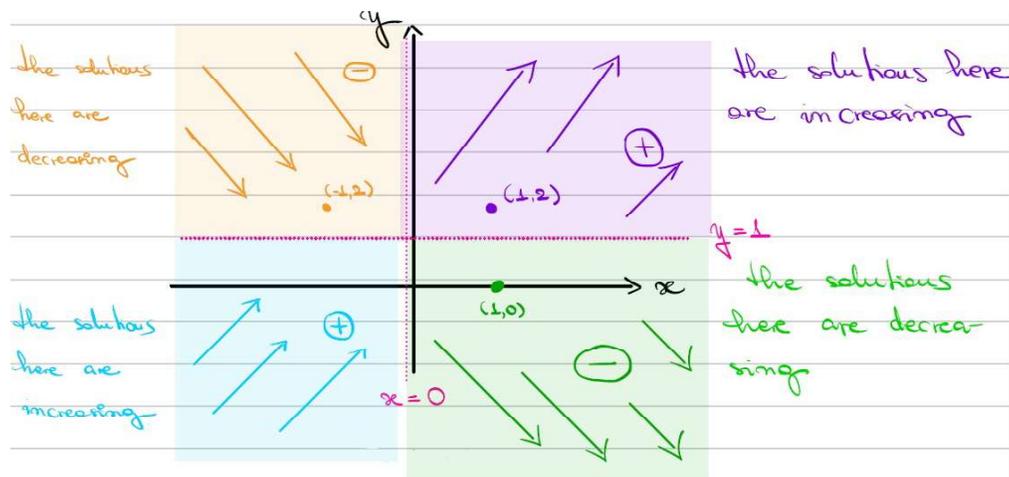


Figure 5: Solutions of the ODE $y' = 2x(y - 1)$.

This information allows us to sketch the slope field of the equation (see once again Figure 5). In each region, we draw short line segments with slopes consistent with the sign of y' . The resulting picture is called the **slope field** of the equation. Observe that the arrows always go from left to right and we will respect that from now on. Notice once again that, in order to sketch the slope field, we do not need to solve the ODE explicitly.



Figure 6: Analyzing separately the factors $2x$ and $y - 1$.

Remark 4.2. Let us observe that, in some cases, it is possible to sketch the slope field by analyzing separately the factors appearing in the differential equation. For instance, let us return to the example $y' = 2x(y - 1)$. We begin by considering the factor $2x$ alone and sketching the corresponding qualitative picture, as shown on the left in Figure 6. Second, we take into account the factor $y - 1$ and analyze it in the same way, as illustrated on the right in Figure 6. By combining the information obtained from both factors and determining the sign of their product in each of the four resulting regions, we recover the slope-field sketch shown in Figure 5 above.

So far, we have seen how to sketch slope fields by determining where solutions are increasing or decreasing. Let us now present additional useful strategies that allow us to analyze slope fields more efficiently, especially for separable equations.

Example 4.3. Consider the first-order ODE

$$y' = \frac{y}{y - x}.$$

Our goal is not to solve this equation explicitly, but rather to analyze the qualitative behavior of its solutions using slope fields. First, we observe that the right-hand side is not defined along the line $y = x$. This line is therefore a “forbidden line” meaning that the solutions cannot cross it as the derivative of y does not exist there. Next, we look for points where the derivative vanishes. Notice that $y' = 0$ when $y = 0$. These two curves, $y = 0$ and $y = x$, divide the plane into four regions (see Figure 7). In each region, the sign of y' is constant and as in Example 4.1 we can determine whether solutions are increasing or decreasing by inspecting the sign of the numerator and denominator:

- ★ If $y > 0$ and $y > x$, then $y' > 0$.
- ★ If $y > 0$ and $y < x$, then $y' < 0$.
- ★ If $y < 0$ and $y > x$, then $y' < 0$.

★ If $y < 0$ and $y < x$, then $y' > 0$.

We can then sketch the slope field as in Figure 7 and obtain a qualitative picture of the solutions of the given ODE. Notice that we could have done the same we did in Remark 4.2 with this equation, too.

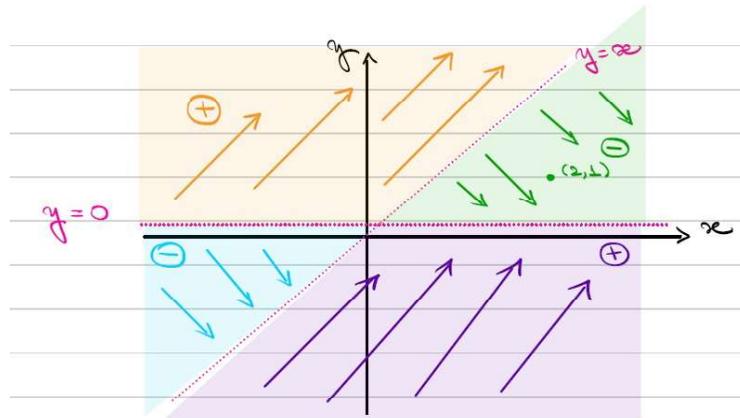


Figure 7: Solutions of the ODE $y' = \frac{y}{y-x}$.

What does the slope field actually look like? When we ask Maple to sketch the slope field of the equation, we obtain a picture such as the one shown in Figure 8. In fact, Maple also allows us to draw several solution curves together with the slope field, as illustrated in Figure 9. Notice that the solution curves follow the directions indicated by the slope field, confirming that the slope field accurately describes the qualitative behavior of the solutions. The same can be done with the equation in Example 4.3 (see Figure 10 and Figure 11).

> DEplot(diff(y(x),x) = 2*x*(y(x) - 1), y(x), x = -2..2, y = -2..2);

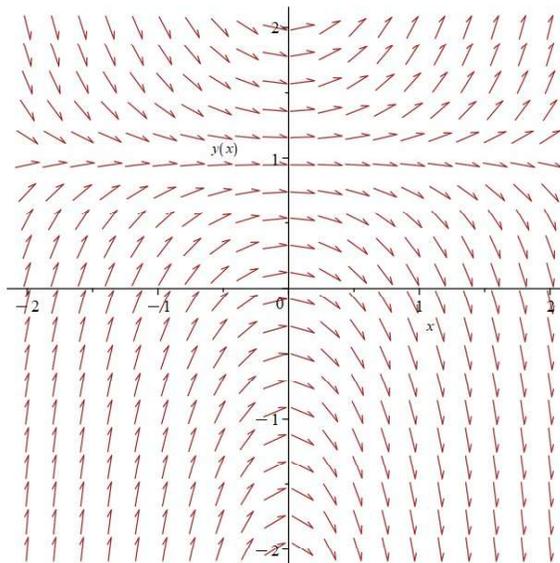


Figure 8: Slope field of the ODE $y' = 2x(y - 1)$.

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> DEplot(diff(y(x),x) = 2*x*(y(x)-1),y(x),x=-2..2,y=-5..5,[[0,1.1],[0,0.5]],linecolor = navy);
```

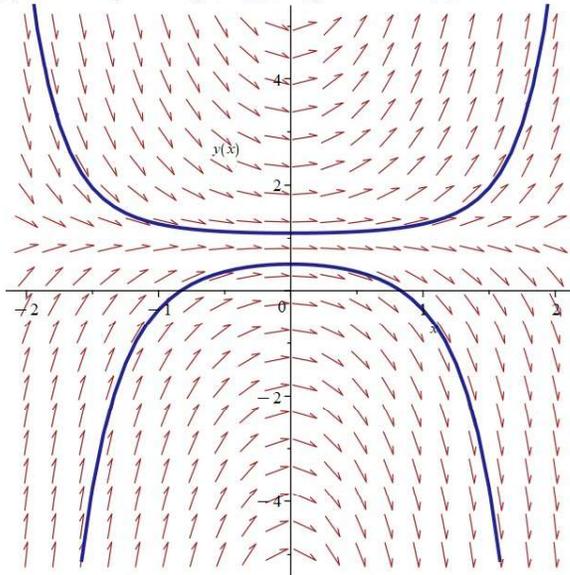


Figure 9: Slope field together with some solutions of the ODE $y' = 2x(y - 1)$.

```
> DEplot(diff(y(x),x) = y(x)/(y(x)-x),y(x),x=-2..2,y=-2..2);
```

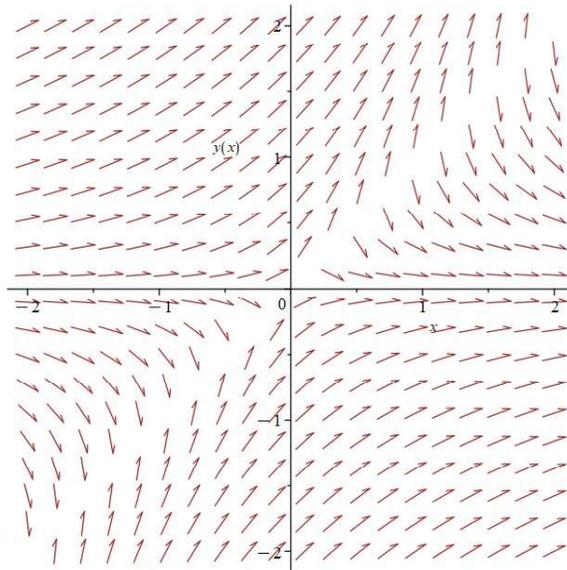


Figure 10: Slope field together with some solutions of the ODE $y' = \frac{y}{y-x}$.

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> DEplot(diff(y(x),x) = -y(x)/(y(x)-x), y(x), x=-2..2, y=-2..2, [[-2,1],[-2,0.2],[-2,-0.2],[-1.1,-2]], linecolor = navy);
Warning, plot may be incomplete, the following error(s) were issued:
cannot evaluate the solution further right of -0.87177970, probably a singularity
Warning, plot may be incomplete, the following error(s) were issued:
cannot evaluate the solution further right of -0.63245547, probably a singularity

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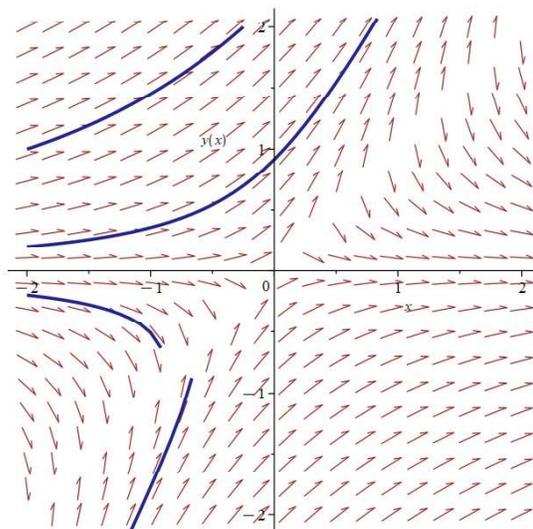


Figure 11: Slope field together with some solutions of the ODE $y' = \frac{y}{y-x}$.

Definition 4.4. A differential equation is called **autonomous** if the free variable does not appear in it, that is, if it can be written as

$$F(y, y', y'', \dots, y^{(n)}) = 0.$$

For instance, the ODE $y' = (1 - y)(1 + y)$ is autonomous (see Example 4.5).

Example 4.5. Consider the autonomous equation (notice it is separable)

$$y' = (1 - y)(1 + y).$$

The stationary solutions are given by $y = \pm 1$. These two horizontal lines divide the plane into three regions. Since the equation is autonomous, the sign of y' depends only on y (see Figure 12).

- ★ For $-1 < y < 1$, we have $y' > 0$, so solutions increase.
- ★ For $y > 1$, we have $y' < 0$, so solutions decrease.
- ★ For $y < -1$, we also have $y' < 0$.

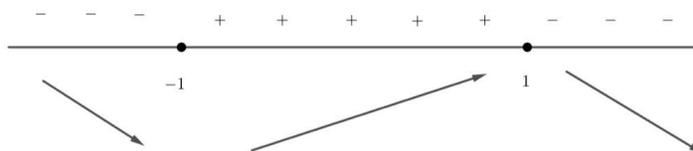


Figure 12: Monotonicity of the equation $y' = (1 - y)(1 + y)$.

Therefore, we can sketch the slope field for this equation in Figure 13.

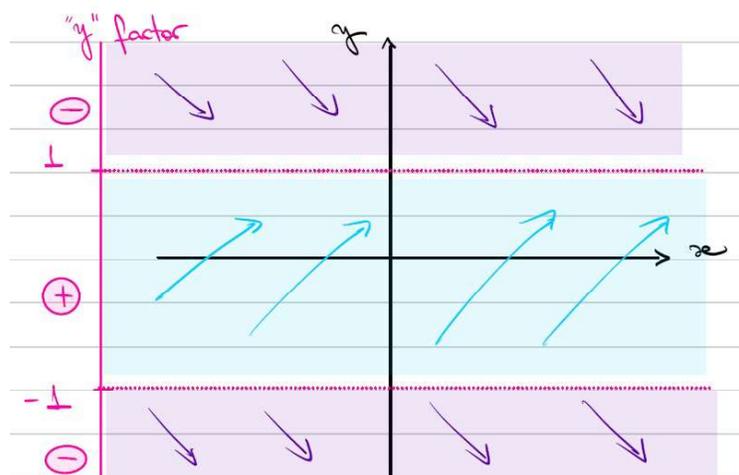


Figure 13: Slope field of the equation $y' = (1 - y)(1 + y)$.

4.3 Stability

Recall that a **stationary solution** (or equilibrium solution as we can see below) of an ODE of the form $y' = f(x, y)$ is a constant function $y(x) = y_0$ such that $f(x, y_0) = 0$ for all x in the interval under consideration. Geometrically, stationary solutions correspond to horizontal lines along which the slope field has zero slope. In the case of an autonomous equation $y' = h(y)$ the stationary solutions are precisely the real numbers y_0 satisfying $h(y_0) = 0$. For instance, in Example 4.5, the stationary solutions are $y(x) = 1$ and $y(x) = -1$ for every $x \in \mathbb{R}$. These solutions play a fundamental role in understanding the qualitative behavior of the differential equation.

Let us now introduce the notion of stability. Roughly speaking, stability describes what happens to solutions that start close to a stationary solution. Notice that the following definition is given only for autonomous equations.

Definition 4.6. Consider a first order autonomous ODE

$$y' = h(y). \tag{9}$$

A number y_0 is called an **equilibrium** of this equation if the constant function $y(x) = y_0$ solves (9). Such an equilibrium is called (asymptotically) **stable** if there exists $d > 0$ such that for every solution $y(x)$ of the equation (9) the following is true: if $|y(x_0) - y_0| < d$ for some $x_0 \in \mathbb{R}$, then $y(x) \rightarrow y_0$ as $x \rightarrow \infty$. Otherwise, we call this equilibrium **unstable**.

Notice that the slope field from Example 4.5 (see Figure 13) shows that the stationary solution $y = 1$ is stable (the arrows from above and below point to this stationary solution) while $y = -1$ is unstable (the arrows go away from this stationary solution). Check Figure 14.

A classical physical example of a stable stationary solution is given by the simple pendulum. If we denote by $\theta(t)$ the angle that the pendulum makes with the vertical, then the vertical downward position $\theta = 0$ corresponds to an equilibrium configuration. Indeed, when the pendulum is at rest at this position,

the net force acting on it is zero. Moreover, this stationary solution is **stable**. If the pendulum is slightly displaced from the vertical position and then released, it does not move away indefinitely. Instead, it oscillates around $\theta = 0$, remaining close to this equilibrium for all time. In the presence of friction or air resistance, these oscillations gradually decrease in amplitude, and the pendulum eventually returns to the vertical position. This shows that $\theta = 0$ is in fact a stable stationary solution.

```
> DEplot(diff(y(x),x) = (1 + y(x))·(1 - y(x)),y(x), x=-2..2,y=-2..2,[[0,0],[0,2],[0,-2],[0,1],[0,-1],[0,1.1],[0,-1.1],[0,0.7],[0,-0.7],[0,0.95],[0,-0.95]],linecolor=navy)
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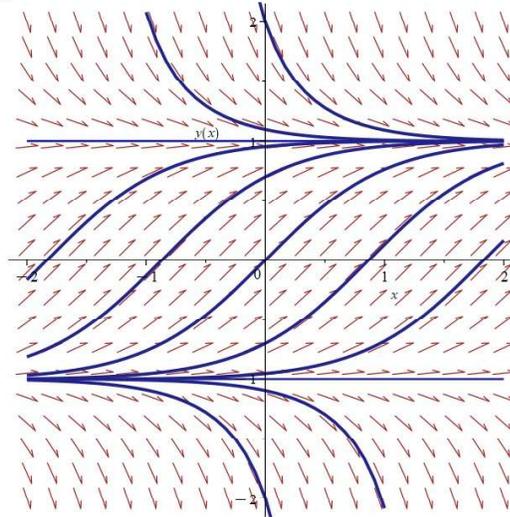


Figure 14: Stability of the stationary solution $y = 1$ of the ODE $y' = (1 - y)(1 + y)$.