5 Lecture #4: Wednesday, February 25th, 2026

5.1 Variation of parameter

We have learned how to solve separable ODEs. As we have already mentioned, there are differential equations that cannot be solved using this method. For instance, the ODE $y' = x + y$ is not separable and therefore cannot be treated with the techniques developed so far. This shows that we need a new approach. In what follows, we will study first-order linear ordinary differential equations. These equations form an important class of ODEs and admit a systematic method of solution. We begin with its formal definition.

Definition 5.1. By a **linear ODE of order 1** we mean any ODE written in the form

$$y' + a(x)y = b(x)$$

where a, b are some functions. This equation is called **homogeneous** if $b(x) = 0$. Given an ODE $y' + a(x)y = b(x)$ by its **associated homogeneous equation** we mean the equation

$$y' + a(x)y = 0.$$

Let us consider our first example.

Example 5.2. Consider the ODE $y' = x + y$. Notice that this equation can be rewritten as $y' - y = x$, which shows that it is a linear ODE of order 1, with coefficients $a(x) = -1$ and $b(x) = x$. In order to solve this equation, we first study the associated homogeneous equation, namely

$$y' - y = 0.$$

This equation can be solved by separation of variables. Indeed, rewriting it in Leibniz notation gives

$$\frac{dy}{dx} = y.$$

If $y \neq 0$, we can separate variables and integrate to obtain

$$\int \frac{dy}{y} = \int dx \Rightarrow \ln|y| = x + C$$

where $C \in \mathbb{R}$ is an arbitrary constant. Exponentiating both sides yields

$$y = \pm e^C \cdot e^x.$$

Absorbing the constant $\pm e^C$ into a single constant, we obtain the general solution

$$y(x) = D \cdot e^x \quad \text{for all } x \in \mathbb{R}.$$

Notice that the stationary solution $y = 0$ is also included in this family by choosing $D = 0$. The second step is to solve the original ODE. To do this, we look for a solution of the form

$$y(x) = D(x) \cdot e^x. \tag{10}$$

Here, the constant D from the homogeneous solution is replaced by a function $D(x)$. This method is known as **variation of constants** (or simply **variation**). Since (10) is assumed to be a solution of the ODE, it must satisfy the equation $y' = x + y$. Substituting $y(x) = D(x)e^x$ into the equation, we obtain

$$D'(x)e^x + D(x)e^x = x + D(x)e^x.$$

Cancelling the common term $D(x)e^x$ on both sides, we are left with

$$D'(x)e^x = x,$$

which implies

$$D'(x) = xe^{-x}.$$

We now integrate this expression. Using integration by parts, we obtain

$$D(x) = \int xe^{-x}dx = -xe^{-x} - e^{-x} + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Therefore, the solution of the original ODE is given by

$$y(x) = D(x)e^x = (-xe^{-x} - e^{-x} + C)e^x = -x - 1 + Ce^x \quad \text{for all } x \in \mathbb{R}.$$

Thus, we conclude that $y(x) = -x - 1 + Ce^x$ is the general solution of the given differential equation.

We now ask whether the procedure followed in Example 5.2 works in general. To this end, consider a first-order linear ODE of the form

$$y' + a(x)y = b(x).$$

Our goal is to understand under which assumptions the method of variation of constants can be applied successfully.

We will look for solutions defined on some interval I. For this reason, we assume that the functions $a(x)$ and $b(x)$ are defined on I. As a first step, we study the associated homogeneous equation

$$y' + a(x)y = 0.$$

Rewriting this equation, and assuming $y \neq 0$, we obtain

$$\int \frac{dy}{y} = \int -a(x)dx.$$

Assuming that $a(x)$ is continuous on I (and therefore integrable on I), we can integrate both sides to get

$$\ln|y| = -A(x) + C,$$

where $A(x)$ is an antiderivative of $a(x)$ and $C \in \mathbb{R}$ is an arbitrary constant. Exponentiating both sides, we obtain

$$y = \pm e^C \cdot e^{-A(x)}.$$

Absorbing the constant $\pm e^C$ into a single constant, we conclude that the general solution of the homogeneous equation is given by

$$y_h(x) = D \cdot e^{-A(x)} \quad \text{for all } x \in I.$$

Notice that this family of solutions includes also the stationary solution $y = 0$ by taking $D = 0$.

In the second step, we look for a solution of the form $y(x) = D(x) \cdot e^{-A(x)}$. Since this function is assumed to satisfy the given ODE, we substitute it into the equation. This yields

$$D'(x)e^{-A(x)} + D(x) \cdot (-A'(x)e^{-A(x)}) + a(x) \cdot D(x) \cdot e^{-A(x)} = b(x).$$

Recalling that $A'(x) = a(x)$, the last two terms cancel each other. Therefore, we are left with

$$D'(x) \cdot e^{-A(x)} = b(x),$$

which implies

$$D'(x) = b(x) \cdot e^{A(x)}.$$

Integrating both sides, we obtain

$$D(x) = \int b(x) \cdot e^{A(x)} dx.$$

For this integral to exist, we must assume that $b(x)$ is continuous on the interval I. Let $B(x)$ denote an antiderivative of the function $b(x) \cdot e^{A(x)}$. Then

$$D(x) = B(x) + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Substituting this expression into the formula for $y(x)$, we conclude that the general solution of the given linear ODE is

$$y(x) = (B(x) + C) \cdot e^{-A(x)} \quad \text{for all } x \in I.$$

This is in fact the proof of the following theorem.

Theorem 5.3 (On solutions of linear ODEs of order 1)**.** Consider a linear ODE $y' + a(x)y = b(x)$. Assume that $a(x)$ and $b(x)$ are continuous functions on an open interval I. Let A be some antiderivative of a on I. Then, the given equation has a solution on I of the form

$$\left(\int b(x)e^{A(x)}dx\right) \cdot e^{-A(x)}.$$

If B is some antiderivative of $b(x)e^{A(x)}$ on I, then a general solution of the given equation on I is

$$y(x) = (B(x) + C)e^{-A(x)}.$$

Example 5.4. Consider the Cauchy problem

$$y' = \frac{y}{x} + 1$$

with the initial condition $y(1) = 1$. First, rewrite the ODE in linear form: $y' - \frac{1}{x}y = 1$. Here $a(x) = -\frac{1}{x}$ and $b(x) = 1$, and we work on an interval I with $x \neq 0$ (in particular, the initial condition suggests that I must be $(0, +\infty)$). Let us solve the associated homogeneous equation

$$y' - \frac{1}{x}y = 0.$$

This gives

$$\frac{dy}{dx} = \frac{1}{x}y,$$

and for $y \neq 0$,

$$\int \frac{dy}{y} = \int \frac{1}{x} dx \Rightarrow \ln|y| = \ln|x| + C$$

Hence, $y_h(x) = Dx$ for $x \in I$.

Now let us apply Variation of constants. We are looking for a solution of the form $y(x) = D(x) \cdot x$. Then

$$y'(x) = D'(x), x + D(x).$$

Substituting into $y' - \frac{1}{x}y = 1$, we get that

$$D'(x) \cdot x + D(x) - \frac{1}{x}(D(x)x) = 1,$$

so the $D(x)$ terms cancel and we get

$$D'(x) \cdot x = 1 \Rightarrow D'(x) = \frac{1}{x}.$$

Integrating, $D(x) = \ln x + C$ (on $x > 0$).

Therefore, the general solution is $y(x) = x(\ln x + C)$. Finally, we impose the initial condition $y(1) = 1$ to get

$$1 = 1 \cdot (\ln 1 + C) = C.$$

Thus, the solution of the Cauchy problem is $y(x) = x(\ln x + 1)$ for $x \in (0, +\infty)$.

Let us observe that the exponential appearing in $y(x) = D(x) \cdot e^{-A(x)}$ in the variation of constants method can be generalized. Suppose that the general solution of the associated homogeneous equation can be written as $y_h(x) = C \cdot u(x)$, where $u(x)$ is a (nonzero) solution of the homogeneous equation. We then look for a solution of the non-homogeneous equation of the form $y(x) = C(x) \cdot u(x)$. Differentiating, we obtain

$$C'(x)u(x) + C(x)u'(x) + a(x)C(x)u(x) = b(x),$$

that is,

$$C'(x)u(x) + C(x) \cdot [u'(x) + a(x)u(x)] = b(x).$$

Since $u(x)$ satisfies the homogeneous equation, we have $u'(x) + a(x)u(x) = 0$, and the previous expression simplifies to

$$C'(x)u(x) = b(x).$$

By having this in mind, the following algorithm makes sense to consider.

Algorithm 5.5 (Variation of parameter for linear ODE of order 1)**.** Let the equation $y' + a(x)y = b(x)$ be given.

(1) Using separation, find a general solution y_h of the associated homogenous equation $y' + a(x)y = 0$. It has the form $y_h(x) = C \cdot u(x)$, which includes also stationary solutions.

(2) Variation of parameter: seek for a solution of the form $y(x) = C(x) \cdot u(x)$. Either substitute this $y(x)$ into the given equation $y' + a(x)y = b(x)$ and cancel, or remember that it leads to the equation $C'(x)u(x) = b(x)$. Then,

$$C(x) = \int \frac{b(x)}{u(x)} dx$$

and we substitute this $C(x)$ into $y(x) = C(x)u(x)$.

(3) If you take for $C(x)$ one particular antiderivative, then you get one particular solution $y_p(x)$, the general solution is then $y = y_p + y_h$. If you include "$+C$" when deriving $C(x)$ then after substituting it into $y(x) = C(x)u(x)$ you get the general solution.

Example 5.6. Let us go back to Example 1.10 above and solve it by variation. Consider the ODE $y' = 2x(y-1)$. Rewrite it in linear form: $y' - 2xy = -2x$. Let us solve first the homogeneous equation $y' - 2xy = 0$ by separation. This gives

$$\int \frac{dy}{y} = \int 2x dx \Rightarrow \ln|y| = x^2 + C.$$

Hence, $y_h(x) = De^{x^2}$. By applying variation of constants now, we look for a solution of the form $y(x) = D(x) \cdot e^{x^2}$. Then, substituting it into the equation $y' - 2xy = -2x$, we get that

$$D'(x)e^{x^2} + D(x)2xe^{x^2} - 2x(D(x)e^{x^2}) = -2x.$$

The $D(x)$ terms cancel, leaving $D'(x)e^{x^2} = -2x$, so $D'(x) = -2xe^{-x^2}$. We now integrate to get

$$D(x) = \int -2xe^{-x^2} dx = e^{-x^2} + C$$

since $(e^{-x^2})' = -2xe^{-x^2}$. Therefore, $y(x) = D(x)e^{x^2} = (e^{-x^2} + C)e^{x^2} = 1 + Ce^{x^2}$. So the general solution is given by

$$y(x) = 1 + Ce^{x^2} \text{ for all } x \in \mathbb{R}$$

as we already knew from Example 1.10, where we solved it by separation.

5.2 Applications of ODEs of order 1 - Logistic growth

Consider the ODE

$$y' = \frac{r}{K}(K - y)y$$

which is known as the **logistic growth equation**. Here, $y(t)$ denotes the size of a population (for instance, a population of rabbits) at time t. In particular, we assume $y(t) \geq 0$ for all t. The parameter $r > 0$ represents the intrinsic growth rate of the population, that is, the growth rate when resources are abundant. The constant $K > 0$ is called the carrying capacity of the environment. It represents the maximum population size that the environment can sustain due to limitations such as food, space and other resources.

The structure of the equation reflects two competing effects. The factor y indicates that population growth is proportional to the current population size, as in the exponential growth equation. On the other hand, the factor $(K - y)$ introduces a limiting effect: when the population is small compared to K, the term $(K - y)$ is close to K and the population grows almost exponentially. As the population approaches the carrying capacity K, the factor $(K - y)$ decreases, slowing down the growth. More precisely, if $y < K$, then $(K - y) > 0$ and therefore $y' > 0$, so the population increases. If $y > K$, then $(K - y) < 0$ and thus $y' < 0$, meaning that the population decreases. Hence, the population is naturally driven toward the equilibrium value $y = K$. In fact, the equation has two stationary solutions: $y = 0$ and $y = K$. As we will see later, the equilibrium $y = 0$ is unstable, since any small population tends to grow, whereas the equilibrium $y = K$ is stable and represents the long-term population level.

Let us now solve the equation. Notice that this is a separable ODE, which can be written as

$$\frac{dy}{dt} = \frac{r}{K}(K - y)y.$$

We already know the stationary solutions $y = 0$ and $y = K$. Assuming that y is different from both, we separate the variables to obtain

$$\int \frac{K}{(K-y)y} dy = \int r dt.$$

The integral on the left-hand side can be computed using partial fractions. Indeed,

$$\frac{K}{(K-y)y} = \frac{1}{y} + \frac{1}{K-y}.$$

Therefore,

$$\int \frac{K}{(K-y)y} dy = \int \frac{1}{y} dy + \int \frac{1}{K-y} dy = \ln|y| - \ln|K-y|.$$

Hence,

$$\ln|y| - \ln|K-y| = rt + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Using logarithmic properties, we obtain

$$\ln\left|\frac{y}{K-y}\right| = rt + C.$$

Exponentiating both sides gives

$$\frac{y}{K-y} = \pm e^C e^{rt}.$$

Writing $D = \pm e^C$, we get

$$y = (K-y)De^{rt}.$$

Solving for y, we arrive at

$$y(t) = \frac{KDe^{rt}}{1 + De^{rt}} \quad \text{for } t \geq 0.$$

Notice that the stationary solution $y = 0$ is included in this family (when $D = 0$), but the stationary solution $y = K$ must be added separately. Therefore, the general solution of the logistic equation is given by

$$y(t) = \frac{KDe^{rt}}{1 + De^{rt}} \quad \text{or} \quad y(t) = K \quad \text{for all } t \geq 0.$$

Finally, let us analyze the behavior as $t \to +\infty$. If $y(t) = K$, the solution is constant. Otherwise, we compute

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \frac{KDe^{rt}}{1 + De^{rt}} == \lim_{t\to+\infty} \frac{KD}{e^{-rt} + D} = K,$$

whenever $D \neq 0$. This confirms that $y = K$ is a stable equilibrium and represents the long-term population size (see Figure 15).

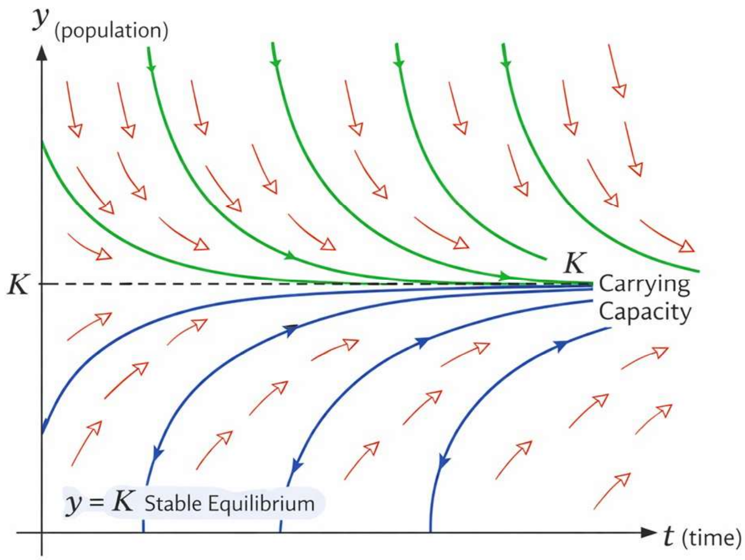

Figure 15: The stationary solution $y = K$ is stable

5.3 Applications of ODEs of order 1 - Free fall with air resistance

Recall that earlier we studied the ODE $y'' = -g$, which models the motion of an object falling under the action of gravity. In that model, $y(t)$ denotes the vertical position of the object, the second derivative $y''(t)$ represents its acceleration and the constant $g > 0$ is the acceleration due to gravity. The negative sign reflects the fact that gravity acts downward. This model assumes that no other forces, such as air resistance, are acting on the object. A more realistic model of free fall must take into account the effect of air resistance (or drag). One commonly used model leads to the ODE

$$y'' = -g + k(y')^2 \tag{11}$$

where $k > 0$ is a constant that depends on physical properties such as the shape of the object and the density of the air. To interpret this equation, recall that $y'(t)$ is the velocity of the object. The term $k(y')^2$ represents the force due to air resistance. In this model, the drag force is proportional to the square of the velocity, which is appropriate for objects moving at relatively high speeds. Since $(y')^2$ is always nonnegative, the term $k(y')^2$ acts in the opposite direction of gravity when the object is falling downward. Physically, this means that gravity accelerates the object downward, while air resistance acts upward, opposing the motion. When the object is moving slowly, the term $(y')^2$ is small and gravity dominates, so the acceleration is close to $-g$. As the speed increases, the air resistance term becomes larger and reduces the net acceleration.

Notice that (11) is a second-order ODE, and we have not yet studied how to solve equations of this type. However, we can reduce the order of the equation by introducing the new variable $v = -y'$. With this change of variables, the equation becomes

$$-v' = -g + kv^2,$$

which is now a first-order ODE. Rewriting this expression, we obtain

$$v' = g - kv^2.$$

Moreover, this can be written as

$$v' = k\left(\frac{g}{k} - v^2\right),$$

which is clearly a separable ODE. Let us first observe that this equation has two stationary solutions given by

$$v(t) = \pm\sqrt{\frac{g}{k}} \quad \text{for all } t \geq 0.$$

If v is not one of these constant solutions, we can proceed by separating variables. This gives

$$\int \frac{1}{\frac{g}{k} - v^2} dv = \int k dt.$$

To compute the integral on the left-hand side, we use partial fractions. For convenience, let

$$a = \sqrt{\frac{g}{k}},$$

so that $a^2 = \frac{g}{k}$. Multiplying both sides of the equation by -1, we obtain

$$\int \frac{1}{v^2 - a^2} dv = \int -k dt.$$

That is,

$$\int \frac{1}{(v-a)(v+a)} dv = -\int k dt.$$

Using partial fractions, we find

$$\int \frac{1}{v-a} dv - \int \frac{1}{v+a} dv = -2\sqrt{gk} \cdot t + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Therefore,

$$\ln|v-a| - \ln|v+a| = -2\sqrt{gk} \cdot t + C,$$

or equivalently,

$$\ln\left|\frac{v-a}{v+a}\right| = -2\sqrt{gk} \cdot t + C.$$

Exponentiating both sides yields

$$\frac{v-a}{v+a} = \pm e^C \cdot e^{-2\sqrt{gk}\cdot t}.$$

Solving for v, we obtain

$$v - a = (v+a) \cdot D \cdot e^{-2\sqrt{gk}\cdot t},$$

where $D = \pm e^C$. Finally, solving this expression for v, we find

$$v(t) = \sqrt{\frac{g}{k}} \cdot \frac{1 + D \cdot e^{-2\sqrt{gk}\cdot t}}{1 - D \cdot e^{-2\sqrt{gk}\cdot t}} \quad \text{for all } t \geq 0.$$

Notice that

$$\lim_{t\to+\infty} v(t) = \sqrt{\frac{g}{k}}.$$

This constant value is called the terminal velocity. Terminal velocity is the maximum constant speed that a falling object reaches when the forces acting on it are in balance, so that its acceleration becomes zero.