# The Daugavet property and its polynomial version

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This talk is based on a recent joint work with

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⋆ Yoël **Perreau**
**Tartu University, Estonia**

⋆ Igor K. **Daugavet** (1932-2022) (Saint Peterburg State University)

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$$\|T\| = 1 = \|\mathrm{Id} + T\| \Rightarrow 1 = \|\mathrm{Id} + T\| \neq 1 + \|T\| = 2.$$

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- ⋆ What results do we have?
- ⋆ Is there more to tell?

# Let's start now?

Not yet!
How about starting with a proof?

## Definition (Almost diffuse operators)

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$$g \in C(K), \|g\|_\infty \leq 1, g(s) = 0, \forall s \notin U(s_0) \Rightarrow \|Tg\| < \varepsilon.$$

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## Result *Prove it!*

If $T : C(K) \to C(K)$ is almost diffuse, then $\|\mathrm{Id} + T\| \geq 1 + \|T\|$.

So, we want to know when

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for every [          ]

bounded linear operator $T$.

# Before that, in what world will we be working?

# (Homogeneous) Polynomials

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$$F(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = F(x_1, \dots, x_N)$$

holds true for every permutation $\sigma$ of $\{1, \dots, N\}$ and for every $N$-tuple $(x_1, \dots, x_N) \in X^N$) such that

$$P(x) = F(x, \dots, x)$$

for every $x \in X$.

# (Homogeneous) Polynomials

⋆ $\mathcal{P}(^N X, Y) =$ $N$-homogeneous polynomials from $X$ into $Y$[2][3].

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- ⋆ $\mathcal{P}(^N X, Y)$ = $N$-homogeneous polynomials from $X$ into $Y$[2][3].
- ⋆ $\mathcal{P}(X, Y)$ = all (continuous) polynomials of the form

$$P = \sum_{k=0}^{m} P_k$$

where $P_k \in \mathcal{P}(^k X, Y)$ for every $k = 0, 1, \ldots, m$.

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- In $\mathcal{P}(X, Y)$, we define

$$\|P\| := \sup_{x \in B_X} \|P(x)\| \quad (P \in \mathcal{P}(X, Y)).$$

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- ⋆ A polynomial $P \in \mathcal{P}(X, Y)$ is said to be **weakly compact** if $P(B_X)$ is a relatively weakly compact[4] subset of $Y$.
- ⋆ A polynomial $P$ is a **rank-one** if $P(x) = p(x)y_0$ for every $x \in X$, where $p \in \mathcal{P}(X)$ and $y_0 \in Y$.

[4] if its closure is weakly compact

# (Homogeneous) Polynomials

(Personal) relevant references about this topic:

- S. **Dineen**, Complex Analysis on infinite dimensional spaces
- J. **Mujica**, Complex analysis in Banach spaces
- P. **Hájek** and M. **Johanis**, Smooth Analysis in Banach spaces

# What property are we considering?

# The Daugavet property

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- ⋆ $C(K)$ with $K$ perfect [**Daugavet**, '63]
- ⋆ $L_1(\mu)$ with $\mu$ atomless [**Lozanovskii**, '66]
- ⋆ Some Banach algebras of holomorphic functions on Banach spaces [**Wojtaszczyk**, '92], [**Werner**, '97], [**Jung**, '23]

# The Daugavet property

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**Recent reference**

V. **Kadets**, M. **Martín**, A. **Rueda Zoca**, D. **Werner**
*Banach spaces with the Daugavet property*
Preprint
2025

## Surprising results

[5] $\forall\ x \neq y \in X$ with $\|x\| = \|y\| = 1$ we have $\left\| \frac{x+y}{2} \right\| < 1$.

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# The Daugavet property

### Surprising results

- ☆ If $X \in$ DPr, then $X$ fails the RNP.
  - ⋆ Actually, every combination of slices of $B_X$ has diameter 2.
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## Still surprising results (the most surprising?)

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**V. Kadets, R. Shvidkoy, G. Sirotkin, D. Werner, TAMS, 2000**

$X$ has the DPr if and only if $B_X$ is equal to the closed convex hull of the set

$$\{y \in B_X : \|x + y\| > 2 - \varepsilon\}$$

for every $x \in B_X$ and $\varepsilon > 0$.

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for every $x \in B_X$ and $\varepsilon > 0$. In other words, for every $x \in S_X$ and every slice[a] $S$ of $B_X$, we have

$$\sup_{y \in S} \|x + y\| = 2.$$

[a] For $x^* \in X^*$ and $\varepsilon > 0$, a **slice of a set** $A$ is a set of the form

$$S(A, x^*, \varepsilon) = \{x \in A : x^*(x) > x^*(A) - \varepsilon\}.$$

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⋆ A lot of results on the DPr can be proved by using this lemma.

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# The polynomial Daugavet property

**Y.S. Choi, D. García, M. Maestre, M. Martín, 2007, Studia Math.**

Let $X$ be a Banach space. TFAE:

(a) $X$ has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall P \in \mathcal{P}(X, \mathbb{K})$ with $\|P\| = 1$, $\exists y \in B_X$ and $\exists \omega \in \mathbb{T}$ (a modulus-one scalar) such that

$$\operatorname{Re} \omega P(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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# What kind of problems do we want to tackle?

# Main problem

$$\boxed{\textit{DPr} \Leftrightarrow \textit{polynomial DPr?}}$$

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(2) However, it is known that in infinite dimensional Banach spaces, the only weakly continuous homogeneous polynomials are those of **finite type**, that is, of the form

$$P(x) = \sum_{j=1}^{m} \alpha_j \varphi_j^N(x)$$

with $\alpha_j \in \mathbb{K}$ and $\varphi_j \in X^*$.
[R. **Aron** and J.B. **Prolla**, 1980]

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[J. **Ferrera**, J. **Gomez Gil** and J.L. **González Llavona**, 1983]

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**Conclusion**

From these results in

(1) [D. **Cabezas**, M. **Martín** and A.M. **Peralta**, 2024]

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we **cannot** expect to get the equivalence between the DPr and the polynomial DPr as a simple consequence of the characterization due to Shvidkoy's.

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**The general strategy used so far**

[7] If every continuous weakly compact operator $T : X \to Y$ transforms weakly compact sets in $X$ into norm-compact sets in $Y$.

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  - and the **Dunford-Pettis property** on $L_1[0,1]$.

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- ⋆ In particular, $\mathcal{F}(\ell_2)$ **fails** the DPP.
- ⋆ It is **not known** whether $\mathcal{F}(\mathbb{R}^n)$ has the DPP. (A. **Procházka** - WhatsApp message)

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(4) Some consequences.

# WHAT RESULTS DO WE HAVE?

# The tool(s)

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The **polynomial-star topology** of $X^{**}$ is the smallest topology on $X^{**}$ for which a net $(x_\alpha)$ in $X^{**}$ converges to a point $x$ in $X^{**}$ if and only if $\hat{p}(x_\alpha) \longrightarrow \hat{p}(x)$ for every scalar-valued polynomial $p$ on $X$, where $\hat{p}$ denotes the Aron-Berner extension[a] of $p$ to $X^{**}$.

[a] [R.M. Aron and P.D. Berner, 1978]

# The tool(s)

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- ⋆ This provides a polynomial-star Goldstine theorem.
- ⋆ We will use the ideas of the proof of this result.

# Our results

**[D., Martín, Perreau, 2025]**
**(Shvidkoy's lemma for the weak polynomial topology)**

Let $X$ be a Banach space with the DPr.

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Let $X$ be a Banach space with the DPr. Then, for every $x \in S_X$ and $y \in B_X$, we can find a net $(y_\alpha) \subseteq B_X$ which converges to $y$ in the weak polynomial topology of $B_X$ and such that $\|x + y_\alpha\| \to 2$.

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**Corollary (DPr $\Leftrightarrow$ polynomial DPr)**

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**Corollary (DPr $\Leftrightarrow$ polynomial DPr)**

A Banach space $X$ with the DPr satisfies the polynomial DPr.

Recall that TFAE:

(a) $X$ has the polynomial DPr.

(b) $\forall x \in S_X$, $\forall \varepsilon > 0$, $\forall p \in \mathcal{P}(X, \mathbb{K})$ with $\|p\| = 1$, $\exists y \in B_X$ and $\exists \omega \in S_{\mathbb{K}}$ such that

$$\operatorname{Re} \omega p(y) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y\| > 2 - \varepsilon.$$

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- ⋆ By doing this, we can find $(y_\alpha) \subset B_X$ converging to $y$ in the weak polynomial topology and such that $\|x + \omega y\| \to 2$.
- ⋆ In particular, $p(y_\alpha) \longrightarrow p(y)$ and so we can find $\alpha$ such that

$$\operatorname{Re} \omega p(y_\alpha) > 1 - \varepsilon \quad \text{and} \quad \|x + \omega y_\alpha\| > 2 - \varepsilon$$

and this exactly what we wanted. ■

# Our results

All we need to do now is to prove the main result.

**[D., Martín, Perreau, 2025]**
**(Shvidkoy's lemma for the weak polynomial topology)**

Let $X$ be a Banach space with the DPr. Then, for every $x \in S_X$ and $y \in B_X$, we can find a net $(y_\alpha) \subseteq B_X$ which converges to $y$ in the weak polynomial topology of $B_X$ and such that $\|x + y_\alpha\| \to 2$.

# Is there more to tell?

# A consequence

- ⋆ This result shows in particular that, for **real** Banach spaces with the DPr, we have
$$\|p\| = \sup_{x \in S_X} |p(x)|$$
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- ⋆ **Real** polynomials can easily attain their norms in the interior of the ball. (the polynomial $x \mapsto 1 - \langle x, x \rangle$ on a Hilbert space)
- ⋆ For a systematic study we refer to [S.D. and R. Medina, 2024] (Here on Tuesday, 29 April 2025)

# Daugavet centers

⋆ $G \in \mathcal{L}(X, Y)$ is a **Daugavet center** if $\|G + T\| = \|G\| + \|T\|$ holds for every rank-one $T \in \mathcal{L}(X, Y)$.
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- ⋆ $X$ has the DPr if and only if $\mathrm{Id}$ is a Daugavet center.
- ⋆ **(Shvidkoy-type lemma for Daugavet centers)** Let $X$ be a Banach space and $G \in \mathcal{L}(X, Y)$ with $\|G\| = 1$ be a Daugavet center. Then for every $x \in B_X$ and $y \in S_Y$, there exists a net $(x_\alpha)$ in $B_X$ such that $x_\alpha \to x$ weakly and $\|y + G(x_\alpha)\| \to 2$.
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Let $X$ be a Banach space and $G \in \mathcal{L}(X, Y)$ be a Daugavet center. For every $x \in B_X$ and $y \in S_Y$, there exists a net $(x_\alpha)$ in $B_X$ such that $x_\alpha \to x$ in the weak polynomial topology and $\|y + G(x_\alpha)\| \to 2$.

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Every (linear) Daugavet center is a polynomial Daugavet center.

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**Corollary**

Every (linear) Daugavet center is a polynomial Daugavet center.

$\star$ The linearity of $G$ above is crucial: we **do not know** if, given a polynomial $Q \in \mathcal{P}(X, Y)$ such that $\|Q + T\| = \|Q\| + \|T\|$ for every rank-one $T \in \mathcal{L}(X, Y)$, we actually have that the same equality holds for $T$ being a rank-one polynomial from $X$ into $Y$.

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⋆ Is there a bidual space with the DPr?

# Thank you very much for your attention!