

3 Practice #1: Wednesday, February 18th, 2026

3.1 Problem 1

Consider the ordinary differential equation

$$y' = \frac{3y}{x-1}.$$

We will solve it by separation of variables. Then, we will find the general solution and solve the following initial value problems:

(a) $y(-1) = -8$,

(b) $y(2) = 8$,

(c) $y(1) = 8$.

We solve the equation

$$y' = \frac{3y}{x-1}$$

by separation of variables. Rewriting it in Leibniz notation, we have

$$\frac{dy}{y} = \frac{3y}{x-1}.$$

Assuming $y \neq 0$, we separate the variables:

$$\frac{1}{y} dy = \frac{3}{x-1} dx.$$

Integrating both sides, we obtain

$$\int \frac{1}{y} dy = \int \frac{3}{x-1} dx,$$

which gives

$$\ln |y| = 3 \ln |x-1| + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Using properties of logarithms, we rewrite this as

$$\ln |y| = \ln |x-1|^3 + C.$$

Exponentiating both sides, we obtain

$$|y| = e^C |x-1|^3.$$

Since $e^C > 0$, we may write $D = \pm e^C$ and obtain the general solution

$$y(x) = D(x-1)^3,$$

valid for $x \neq 1$. Notice that the stationary solution $y(x) = 0$ for every $x \neq 1$ is included by taking $D = 0$.

We now determine the particular solutions.

(a) If $y(-1) = -8$, then

$$-8 = D(-1 - 1)^3 = D(-2)^3 = -8D,$$

so $D = 1$. Therefore,

$$y(x) = (x - 1)^3$$

for every $x \in (-\infty, 1)$.

(b) If $y(2) = 8$, then

$$8 = D(2 - 1)^3 = D,$$

so $D = 8$. Hence,

$$y(x) = 8(x - 1)^3$$

for every $x \in (1, +\infty)$.

(c) If $y(1) = 8$, we observe that the differential equation is not defined at $x = 1$, since the right-hand side $\frac{3y}{x-1}$ is undefined there. Therefore, no solution can pass through a point with $x = 1$. Consequently, the initial value problem $y(1) = 8$ has no solution.

Let us consider the right branch of the solution

$$y(x) = D(x - 1)^3, \quad x \in (1, +\infty).$$

We are interested in understanding how this solution behaves as time passes, that is, as $x \rightarrow +\infty$. This behavior is referred to as the **asymptotic rate of growth at** $+\infty$. For large values of x , we observe that $x - 1 \sim x$, and therefore

$$y(x) = D(x - 1)^3 \sim Dx^3 \quad \text{as } x \rightarrow +\infty.$$

In other words, the solution grows cubically at infinity. This means that, for large x , the dominant term governing the behavior of the solution is Dx^3 .

3.2 Problem 2

Let us consider now the ordinary differential equation

$$2y' = \frac{e^x}{y}.$$

We will find the general solution and solve the following initial value problems:

(a) $y(0) = 1$,

(b) $y(0) = -1$,

(c) $y(1) = 0$,

(d) $y(1) = 1$.

We start with the equation

$$2y' = \frac{e^x}{y}.$$

Notice first that y must be different from 0. Rewriting the equation in Leibniz notation, we obtain

$$2\frac{dy}{dx} = \frac{e^x}{y}.$$

Multiplying both sides by y , we get

$$2y\frac{dy}{dx} = e^x.$$

Now we separate the variables:

$$2y dy = e^x dx.$$

Integrating both sides, we obtain

$$\int 2y dy = \int e^x dx,$$

which gives

$$y^2 = e^x + C,$$

where $C \in \mathbb{R}$ is an arbitrary constant. Therefore, the general solution is

$$y(x) = \pm\sqrt{e^x + C}$$

defined for those x such that $e^x + C > 0$ (recall that $y \neq 0$). Notice that $y = 0$ is not a solution of the differential equation, since the right-hand side is not defined at $y = 0$.

We now solve the initial value problems.

(a) If $y(0) = 1$, then

$$1 = \pm\sqrt{e^0 + C} = \pm\sqrt{1 + C}.$$

Since the value is positive, we take the positive branch and obtain

$$1 = \sqrt{1 + C} \quad \Rightarrow \quad C = 0.$$

Thus, the solution is

$$y(x) = \sqrt{e^x} = e^{\frac{x}{2}}$$

for every $x \in \mathbb{R}$.

(b) If $y(0) = -1$, then

$$-1 = \pm\sqrt{1 + C}.$$

Since the value is negative, we take the negative branch and again obtain $C = 0$. Hence,

$$y(x) = -\sqrt{e^x} = -e^{\frac{x}{2}}$$

for every $x \in \mathbb{R}$.

(c) If $y(1) = 0$, then

$$0 = \pm\sqrt{e + C} \quad \Rightarrow \quad e + C = 0 \quad \Rightarrow \quad C = -e.$$

This would formally give

$$y(x) = \pm\sqrt{e^x - e}$$

for every $x \in \mathbb{R}$ such that $e^x - e > 0$, that is, $e^x > e$, that is, $x > 1$. This means that we would have the solution $y(x) = \pm\sqrt{e^x - e}$ for every $x \in (1, +\infty)$, which does not include the point $x = 1$. Therefore,

there is no solution in this case. In fact, the differential equation is not defined at $y = 0$ as we have seen before, and therefore no solution of the equation can pass through a point where $y = 0$. Consequently, the initial value problem $y(1) = 0$ has no solution.

(d) If $y(1) = 1$, then

$$1 = \pm\sqrt{e + C}.$$

Since the value is positive, we take the positive branch and obtain

$$1 = \sqrt{e + C} \Rightarrow e + C = 1 \Rightarrow C = 1 - e.$$

Therefore, the solution is

$$y(x) = \sqrt{e^x + 1 - e},$$

defined for those x such that $e^x + 1 - e > 0$, which is the same as $e^x > e - 1$. That is, $y(x) = \sqrt{e^x + 1 - e}$ for every $x > \ln(e - 1)$.

3.3 Problem 3

In many practical problems arising in Physics and other applied sciences, derivatives with respect to time are often denoted using the dot notation instead of the prime notation. Thus, if $x = x(t)$ is a function of time, we write

$$\dot{x} = \frac{dx}{dt}.$$

Consider the ordinary differential equation

$$\dot{x} = 2tx^2$$

where $x = x(t)$. We will find the general solution and solve the following initial value problems. In each case, as the other examples, we will determine the region of validity of the solution.

(a) $x(-2) = 1$,

(b) $x(-1) = -1$,

(c) $x(1) = -\frac{1}{2}$,

(d) $x(3) = -\frac{1}{4}$,

(e) $x(1) = 0$.

We start from

$$\dot{x} = 2tx^2,$$

that is,

$$\frac{dx}{dt} = 2tx^2.$$

Assuming $x \neq 0$, we separate the variables:

$$\frac{1}{x^2} dx = 2t dt.$$

Integrating both sides, we obtain

$$\int x^{-2} dx = \int 2t dt,$$

which gives

$$-\frac{1}{x} = t^2 + C$$

where $C \in \mathbb{R}$ is arbitrary. Rewriting this expression, we obtain

$$\frac{1}{x} = -t^2 - C.$$

Renaming the constant, we may write

$$\frac{1}{x} = C - t^2,$$

and therefore the general solution is

$$x(t) = \frac{1}{C - t^2}$$

defined for all t such that $C - t^2 \neq 0$. Notice also that $x = 0$ is a stationary solution of the differential equation, since substituting $x = 0$ into $\dot{x} = 2tx^2$ gives $0 = 0$. Hence, $x(t) = 0$ is also a solution. Notice that, in this case, we cannot combine the stationary solution $x(t) = 0$ with the family $x(t) = \frac{1}{C-t^2}$ into a single formula describing the general solution, since the latter expression is not defined at $x = 0$ and does not include this constant solution.

We now determine the particular solutions.

(a) If $x(-2) = 1$, then

$$1 = \frac{1}{C - (-2)^2} = \frac{1}{C - 4},$$

so $C - 4 = 1$ and therefore $C = 5$. Hence,

$$x(t) = \frac{1}{5 - t^2}.$$

The denominator vanishes when $t = \pm\sqrt{5}$. Since the initial time is $t = -2$ and $-2 \in (-\sqrt{5}, \sqrt{5})$, the region of validity is $t \in (-\sqrt{5}, \sqrt{5})$.

(b) If $x(-1) = -1$, then

$$-1 = \frac{1}{C - 1},$$

so $C - 1 = -1$ and therefore $C = 0$. Thus,

$$x(t) = \frac{1}{-t^2} = -\frac{1}{t^2}.$$

This expression is defined for $t \neq 0$. Since the initial time is $t = -1$, the maximal interval containing -1 is $t \in (-\infty, 0)$.

(c) If $x(1) = -\frac{1}{2}$, then

$$-\frac{1}{2} = \frac{1}{C - 1},$$

so $C - 1 = -2$ and therefore $C = -1$. Hence,

$$x(t) = \frac{1}{-1 - t^2} = -\frac{1}{1 + t^2}.$$

Since $1 + t^2 > 0$ for all t , this solution is defined for all $t \in \mathbb{R}$.

(d) If $x(3) = -\frac{1}{4}$, then

$$-\frac{1}{4} = \frac{1}{C - 9},$$

so $C - 9 = -4$ and therefore $C = 5$. Thus,

$$x(t) = \frac{1}{5 - t^2}.$$

The denominator vanishes at $t = \pm\sqrt{5}$. Since $3 > \sqrt{5}$, the maximal interval containing 3 is $t \in (\sqrt{5}, +\infty)$.

(e) If $x(1) = 0$, we observe that the function $x(t) = 0$ satisfies the differential equation for all t . Therefore, the unique solution in this case is

$$x(t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

3.4 Problem 4

Let us solve the Cauchy problem

$$y' = \frac{1 - y^2}{1 - x^2}, \quad y(0) = 0.$$

We start from the differential equation

$$y' = \frac{1 - y^2}{1 - x^2}.$$

This is a separable equation. Rewriting it in Leibniz notation, we obtain

$$\frac{dy}{dx} = \frac{1 - y^2}{1 - x^2}.$$

We have two stationary solutions: $y(x) = \pm 1$ for every $x \neq -1$ and $x \neq 1$. Assuming $y \neq \pm 1$, we separate the variables:

$$\frac{1}{1 - y^2} dy = \frac{1}{1 - x^2} dx.$$

Integrating both sides, we get

$$\int \frac{1}{1 - y^2} dy = \int \frac{1}{1 - x^2} dx.$$

Recall that (we will solve this integral by the end of the exercise)

$$\int \frac{1}{1 - z^2} dz = \frac{1}{2} \ln \left| \frac{1 + z}{1 - z} \right|. \quad (8)$$

Therefore,

$$\frac{1}{2} \ln \left| \frac{1+y}{1-y} \right| = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C,$$

where $C \in \mathbb{R}$ is arbitrary. Multiplying by 2, we obtain

$$\ln \left| \frac{1+y}{1-y} \right| = \ln \left| \frac{1+x}{1-x} \right| + C.$$

Exponentiating both sides, we get

$$\frac{1+y}{1-y} = D \frac{1+x}{1-x},$$

where $D = \pm e^C$.

We now determine D using the initial condition $y(0) = 0$. Substituting $x = 0$ and $y = 0$, we obtain

$$\frac{1+0}{1-0} = D \frac{1+0}{1-0},$$

that is,

$$1 = D.$$

Hence, $D = 1$ and

$$\frac{1+y}{1-y} = \frac{1+x}{1-x}.$$

Solving this equation for y , we obtain

$$(1+y)(1-x) = (1-y)(1+x).$$

Expanding both sides,

$$1 - x + y - xy = 1 + x - y - xy.$$

Simplifying,

$$-x + y = x - y,$$

which gives

$$2y = 2x,$$

and therefore

$$y(x) = x.$$

Finally, we observe that the differential equation is not defined at $x = \pm 1$. Since the initial condition is given at $x = 0$, the maximal interval of validity of the solution is $x \in (-1, 1)$. Thus, the solution of the Cauchy problem is

$$y(x) = x \quad \text{for } x \in (-1, 1).$$

Let us now solve the integral (8). First, we factor the denominator:

$$1 - z^2 = (1 - z)(1 + z).$$

We use partial fractions and write

$$\frac{1}{1 - z^2} = \frac{A}{1 - z} + \frac{B}{1 + z}.$$

Multiplying both sides by $(1 - z)(1 + z)$, we obtain

$$1 = A(1 + z) + B(1 - z).$$

Expanding,

$$1 = A + Az + B - Bz = (A + B) + (A - B)z.$$

By identifying coefficients, we get

$$A + B = 1, \quad A - B = 0.$$

Solving this system gives

$$A = \frac{1}{2}, \quad B = \frac{1}{2}.$$

Therefore,

$$\frac{1}{1 - z^2} = \frac{1}{2} \left(\frac{1}{1 - z} + \frac{1}{1 + z} \right).$$

We now integrate:

$$\int \frac{1}{1 - z^2} dz = \frac{1}{2} \int \left(\frac{1}{1 - z} + \frac{1}{1 + z} \right) dz.$$

We compute each term:

$$\int \frac{1}{1 - z} dz = -\ln |1 - z| \quad \text{and} \quad \int \frac{1}{1 + z} dz = \ln |1 + z|.$$

Hence,

$$\int \frac{1}{1 - z^2} dz = \frac{1}{2} (-\ln |1 - z| + \ln |1 + z|) + C.$$

Finally, combining the logarithms,

$$\int \frac{1}{1 - z^2} dz = \frac{1}{2} \ln \left| \frac{1 + z}{1 - z} \right| + C.$$