6 Practice #2: Wednesday, February 25th, 2026

6.1 Problem 1

In today's practice class, we will see linear ordinary differential equations of first order, that is, equations of the form

$$y' + a(x)y = b(x),$$

where $a(x)$ and $b(x)$ are given functions on some interval I.

Example 1. Consider the ODE

$$2y' = \frac{e^x}{y}.$$

Dividing both sides by 2, we obtain

$$y' = \frac{e^x}{2y}.$$

The right-hand side contains the term $\frac{1}{y}$, which is not linear in y. Since a linear equation must be of the form $y' + a(x)y = b(x)$, this equation cannot be written in that form. Therefore, this equation is **not linear**.

Example 2. Consider the ODE

$$y' = \frac{3y}{x-1}.$$

We can rewrite it as

$$y' - \frac{3}{x-1}y = 0.$$

This is a linear equation of first order with

$$a(x) = -\frac{3}{x-1} \quad \text{and} \quad b(x) = 0.$$

Since $b(x) = 0$, the equation is called homogeneous. Moreover, it can be solved by separation of variables because it can be written in the form

$$\frac{dy}{dx} = \frac{3}{x-1}y,$$

which is of the type $y' = g(x)h(y)$. We will not solve it here, since this has already been done in the previous section.

Example 3. Consider now the equation

$$y' - \cos(x)y = -\cos(x).$$

This is a linear ODE of the form $y' + a(x)y = b(x)$ with

$$a(x) = -\cos(x) \quad \text{and} \quad b(x) = -\cos(x).$$

Notice also that it can be rewritten as

$$y' = \cos(x)(y-1),$$

which shows that it is separable. However, we solve it here using the method of variation of constants.

First, we consider the associated homogeneous equation

$$y' - \cos(x)y = 0,$$

that is,

$$y' = \cos(x)y.$$

Notice that $y(x) = 0$ for every $x \in \mathbb{R}$ is a stationary solution. Now, assuming that $y \neq 0$, we separate the variables as follows

$$\frac{1}{y}\,dy = \cos(x)\,dx.$$

Integrating both sides, we obtain

$$\ln|y| = \sin(x) + C,$$

and therefore the general solution of the homogeneous equation is

$$y_h(x) = De^{\sin(x)},$$

where $D \in \mathbb{R}$, which includes the stationary solution $y = 0$ by taking $D = 0$.

We now look for a solution of the original equation of the form

$$y(x) = D(x)e^{\sin(x)}.$$

Differentiating, we get

$$y'(x) = D'(x)e^{\sin(x)} + D(x)e^{\sin(x)}\cos(x).$$

Substituting into the equation $y' - \cos(x)y = -\cos(x)$ yields

$$D'(x)e^{\sin(x)} + D(x)e^{\sin(x)}\cos(x) - \cos(x)D(x)e^{\sin(x)} = -\cos(x).$$

The terms involving $D(x)$ cancel, and we obtain

$$D'(x)e^{\sin(x)} = -\cos(x).$$

Hence,

$$D'(x) = -\cos(x)e^{-\sin(x)}.$$

Integrating and using the substitution $u = \sin(x)$, $du = \cos(x)\,dx$, we obtain

$$D(x) = \int -\cos(x)e^{-\sin(x)}\,dx = \int -e^{-u}\,du = e^{-u} + C = e^{-\sin(x)} + C.$$

Finally,

$$y(x) = D(x)e^{\sin(x)} = \left(e^{-\sin(x)} + C\right)e^{\sin(x)} = 1 + Ce^{\sin(x)}.$$

Therefore, the general solution is

$$y(x) = 1 + Ce^{\sin(x)} \quad \text{for all } x \in \mathbb{R}.$$

6.2 Problem 2

Consider the differential equation

$$y' = (x-2)(\ln(x) - y),$$

defined for $x > 0$. We will not attempt to solve this equation analytically. Instead, we study its slope field in order to understand the qualitative behavior of its solutions. First, notice that the derivative is a product of two factors $(x-2)$ and $(\ln(x) - y)$. Therefore, the sign of y' depends on the sign of $(x-2)$ and the sign of $(\ln(x) - y)$. Notice that the derivative always exists and it is zero when $x = 2$ or $\ln(x) = y$. This separates the plan into 4 regions as in Figure 16.

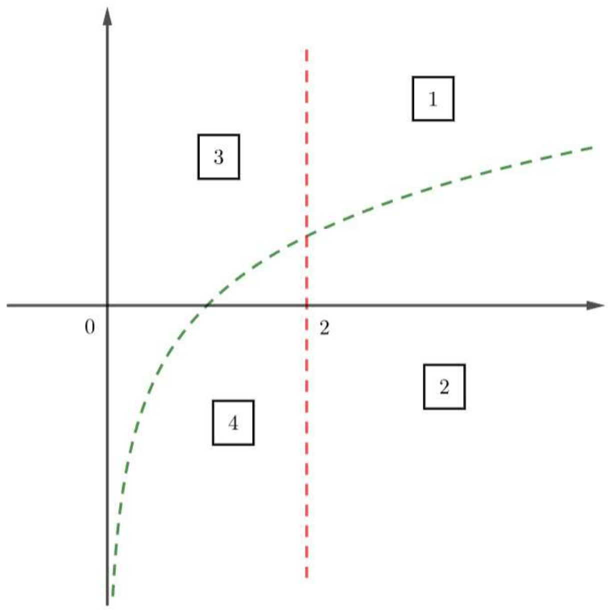

Figure 16: This equation slips the plan into 4 different regions.

We now analyze the slope at the given points.

⋆ At the point $(3, 0)$ we have

$$y' = (3-2)(\ln 3 - 0).$$

Since $3 - 2 > 0$ and $\ln 3 > 0$, we obtain $y' > 0$. Therefore, solutions are increasing near $(3, 0)$.

⋆ At the point $(1, -1)$ we have

$$y' = (1-2)(\ln 1 - (-1)).$$

Since $1 - 2 < 0$ and $\ln 1 = 0$, this becomes

$$y' = (-1)(1) < 0.$$

Hence, solutions are decreasing near $(1, -1)$.

⋆ At the point $(1, 2)$ we have

$$y' = (1-2)(\ln 1 - 2).$$

Again $1 - 2 < 0$ and $\ln 1 = 0$, so

$$y' = (-1)(-2) > 0.$$

Thus, solutions are increasing near $(1, 2)$.

$\star$ At the point $(3, 2)$ we have

$$y' = (3 - 2)(\ln 3 - 2).$$

Since $\ln 3 < 2$, we obtain $y' < 0$. Therefore, solutions are decreasing near $(3, 2)$.

Therefore, we have that (see Figure 17)

- If $x > 2$ and $y < \ln(x)$, then both factors are positive and $y' > 0$.
- If $x > 2$ and $y > \ln(x)$, then $y' < 0$.
- If $0 < x < 2$ and $y < \ln(x)$, then $y' < 0$.
- If $0 < x < 2$ and $y > \ln(x)$, then $y' > 0$.

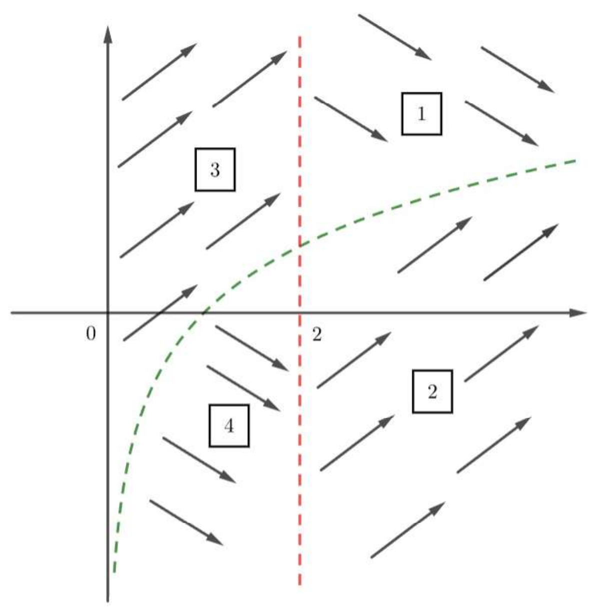

Figure 17: Slope field.

There is another approach that one can use to get the same result. We can analyze the factors separately. In the case of $x - 2$, we have the behavior as in Figure 18.

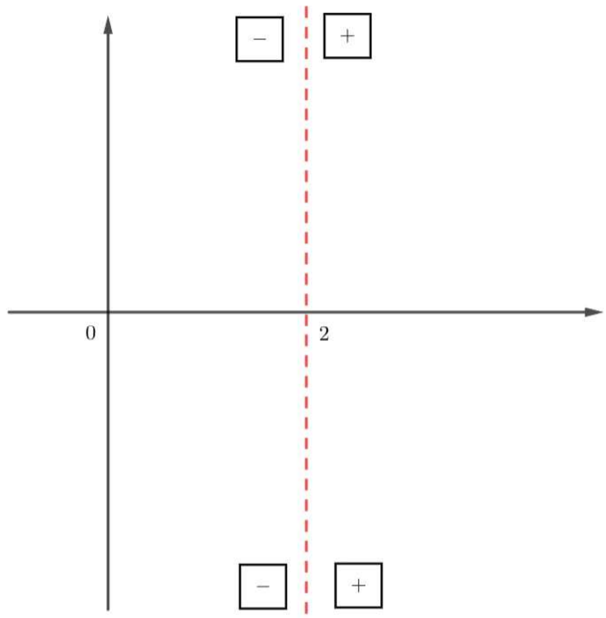

Figure 18: Analyzing only the factor $x - 2$.

Analyzing the second factor $\ln(x) - y$ we have the behavior described in Fiure 19.

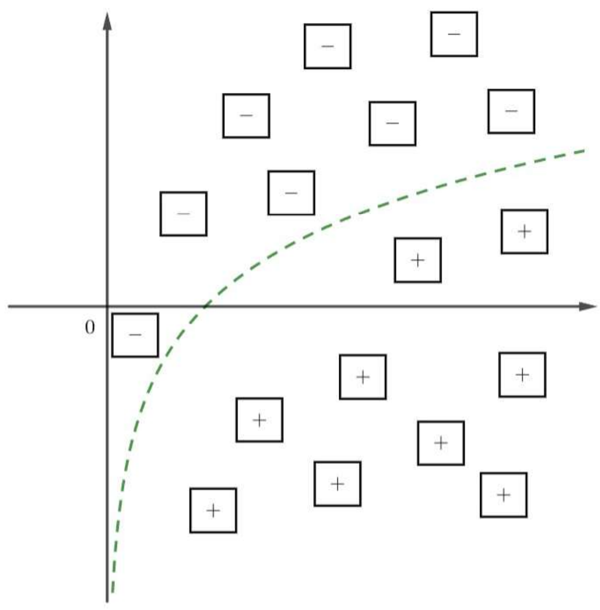

Figure 19: Analyzing only the factor $\ln(x) - y$.

After that we can merge both information to get once again Figure 17 above. Notice that this equation has no stationary solutions.

6.3 Problem 3

Consider the autonomous differential equation

$$y' = \frac{y^2}{1-y}.$$

We will study its slope field, stationary solutions and stability.

Since the right-hand side depends only on y, the equation is autonomous and its slope field is invariant under horizontal translations. In particular, the sign of y' depends only on the value of y, and therefore the qualitative behavior of solutions can be analyzed by studying the function

$$h(y) = \frac{y^2}{1-y}.$$

The expression $\frac{y^2}{1-y}$ is not defined at $y = 1$. Therefore, the horizontal line $y = 1$ is a forbidden line: solutions cannot cross it, and any solution must remain either in the region $y < 1$ or in the region $y > 1$.

A stationary solution of an autonomous equation $y' = h(y)$ is a constant value y_0 such that $h(y_0) = 0$. Here,

$$h(y) = 0 \iff \frac{y^2}{1-y} = 0 \iff y^2 = 0,$$

so the only stationary solution is

$$y(t) = 0 \quad \text{for all } t.$$

Notice that $y = 1$ is not a stationary solution, since the equation is not defined there.

Since $y^2 \geq 0$ for all y, the sign of y' is determined by the sign of the denominator $1 - y$ (see Figure 20).

- If $y < 1$, then $1 - y > 0$ and $y^2 \geq 0$, hence

  $$y' = \frac{y^2}{1-y} \geq 0,$$

  and moreover $y' > 0$ for all $y \neq 0$. Thus, solutions are increasing for $y < 1$, except at the equilibrium $y = 0$ where the slope is zero.

- If $y > 1$, then $1 - y < 0$ and $y^2 > 0$, hence

  $$y' = \frac{y^2}{1-y} < 0.$$

  Thus, solutions are strictly decreasing for $y > 1$.

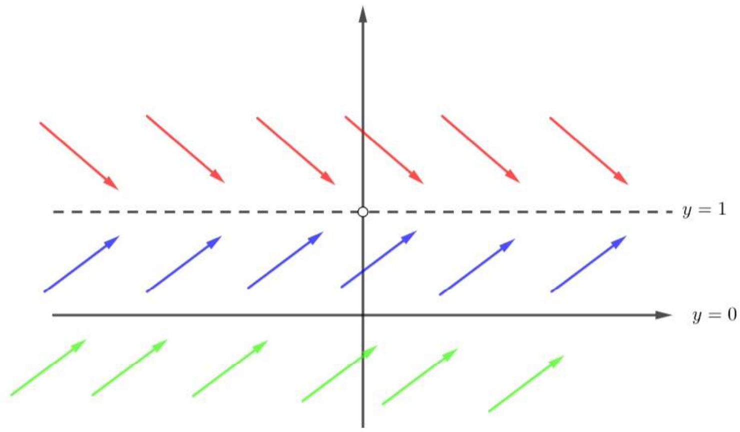

Figure 20: Slope field.

To determine the stability of $y_0 = 0$, we examine solutions starting near 0. If $0 < y_0 < 1$, then $y' > 0$ and the solution increases. In particular, it moves away from 0 as time increases as in Figure 20. If $y_0 < 0$, then again $y' > 0$, so the solution increases. In this case, the solution moves upward toward 0 (from below) as in Figure 20. Therefore, the equilibrium $y = 0$ is unstable.

What is we have considered the autonomous ODE

$$y' = y^2(1 - y).$$

instead? Notice that the solutions would have the same behavior as Figure 20 and then the stationary solution $y = 1$, starting sufficiently close to $y = 1$, are pushed toward $y = 1$ from both sides. Therefore, the stationary solution $y = 1$, in this case, is stable.

Notice also that the previous equation $y' = \frac{y^2}{1-y}$ is separable. Then, if $y \neq 0$, then

$$\frac{dy}{dx} = \frac{y^2}{1-y} \Rightarrow \int \frac{1-y}{y^2} dy = \int dx.$$

Since

$$\int \frac{1-y}{y^2} dy = \int \frac{1}{y^2} dy - \int \frac{1}{y} dy = -\frac{1}{y} - \ln|y|$$

we have that

$$-\frac{1}{y} - \ln|y| = x + C$$

where $C \in \mathbb{R}$ is an arbitrary real number. Notice that we cannot proceed as we cannot solve this last equation for y. Therefore, there is no way to solve this by separation. Notice also that this is not a linear equation of the form $y' + a(x)y = b(x)$.

6.4 Problem 4

Consider the differential equation

$$y' = \frac{2xy}{x^2 - 1}.$$

We study its slope field qualitatively by analyzing where the derivative is zero, where it is not defined, and where it is positive or negative.

The right-hand side is not defined when $x^2 - 1 = 0$, that is, when $x = \pm 1$. Therefore, the vertical lines $x = -1$ and $x = 1$ are forbidden lines: the slope field is not defined there, and solutions cannot cross these lines. Hence, any solution must be contained in one of the three intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, +\infty)$.

We construct in Figure 21 the slope field directly by analyzing the signs separately.

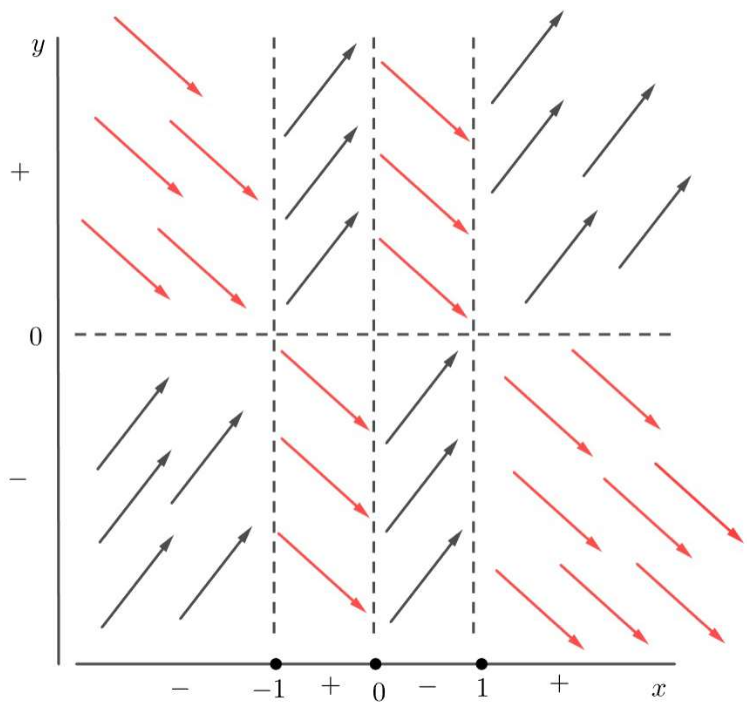

Figure 21: Slope field.

Even though the line $y = 0$ is a stationary solution, the equation is not autonomous because the right-hand side depends explicitly on x. For this reason, the notion of stability of equilibria, as we defined it, does not apply here: stability was introduced only for autonomous equations of the form $y' = h(y)$.