DEN: ODE – theoretical view: separable equations

Definition.
By an explicit **ordinary differential equation of order** n (ODE) we mean any equation of the form

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}),$$

where f is a function of n variables.
By its **solution on an (open) interval** I we mean any function $y = y(x)$ that has all derivatives up to order n on I and for all $x \in I$ satisfies $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$.

If the set of all solutions of a given ODE on some interval I can be expressed using one formula with parameters, we say that this formula is a **general solution** of this ODE.
An individual solution of this equation is called a **particular solution**.

Definition.
Consider an ODE of order n $\quad y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$.
By an **Initial Value Problem** (IVP) or a **Cauchy problem** for this equation we mean any problem of the form
(1) ODE: $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$;
(2) **initial conditions**:
$y(x_0) = y_0$, $y'(x_0) = y_1$, ..., $y^{(n-1)}(x_0) = y_{n-1}$,
here x_0, y_0, y_1, ..., y_{n-1} are some fixed real numbers.

Definition.
By a **separable ODE** we mean any ODE that can be expressed in the form $y' = g(x)h(y)$ for some functions g, h.

Theorem. (existence)
Consider a separable ODE $y' = g(x)h(y)$. Assume that g is continuous on some open interval I and h is continuous on some open interval J. If $h \neq 0$ on J then there is a solution to the given equation on I. Let $G(x)$ be an antiderivative of $g(x)$ on I and $H(y)$ be an antiderivative of $\frac{1}{h(y)}$ on J. If H has an inverse function H_{-1}, then a general solution of the given equation can be expressed as
$y(x) = H_{-1}(G(x) + C)$.
It is valid on intervals I for which $G(x) + C \in D_{H_{-1}}$.

Fact.
Consider a separable ODE $y' = g(x)h(y)$. If y_0 satisfies $h(y_0) = 0$, then the constant function $y(x) = y_0$ is a solution to the given ODE on any open interval $I \subseteq D(g)$ (so-called **stationary solution**).

Algorithm (solving separable ODE by separation)**.**
Given: a differential equation that can be written as $y' = g(x) \cdot h(y)$.
1. Write the equation as $\frac{dy}{dx} = g(x)h(y)$. Move all x (including dx) to the right and all y (including dy) to the left, add integration signs:

$$\frac{dy}{dx} = g(x)h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

2. Explore the possibility that $h(y) = 0$, leading to possible stationary solutions.
3. Assuming that $h(y) \neq 0$, integrate both sides.

$$\int \frac{dy}{h(y)} = \int g(x)\,dx \implies H(y) = G(x) + C.$$

4. If possible, express y as a function of x:

$$H(y) = G(x) + C \implies y(x) = H_{-1}(G(x) + C).$$

5. Exploring the given equation and the solution we obtained, determine conditions of its validity.
6. If an initial condition is given, determine the corresponding particular solution and also the maximal interval of its validity.

Definition.
By a **linear ODE of order 1** we mean any ODE in the form $y' + a(x)y = b(x)$, where a, b are some functions.
This equation is called **homogeneous** if $b(x) = 0$.
Given an ODE $y' + a(x)y = b(x)$, by its **associated homogeneous equation** we mean the equation $y' + a(x)y = 0$.

Theorem. (on **solution** of linear ODE of order 1)
Consider a linear ODE $y' + a(x)y = b(x)$. Assume that $a(x), b(x)$ are continuous functions on some interval I, let A be some antiderivative of a on I. Then the given equation has a solution on I of the form $\Big(\int b(x)e^{A(x)}dx\Big)e^{-A(x)}$.
If B is some antiderivative to $b(x)e^{A(x)}$ on I, then we have the following general solution of the given equation on I:

$$y(x) = (B(x) + C)\,e^{-A(x)}.$$

Algorithm (variation of parameter method for linear ODE of order 1)**.**
Given: equation $y' + a(x)y = b(x)$.
1. Using separation, find a general solution y_h of the associated homogeneous equation $y' + a(x)y = 0$. It has the form $y_h(x) = C \cdot u(x)$, which includes also stationary solutions.
2. Variation of parameter: Seek a solution of the form $y(x) = C(x) \cdot u(x)$.
Either substitute this $y(x)$ into the given equation $y' + a(x)y = b(x)$ and cancel, or remember that it leads to the equation $C'(x)u(x) = b(x)$. Then $C(x) = \int \frac{b(x)}{u(x)}\,dx$, substitute this $C(x)$ into $y(x) = C(x)u(x)$.
3. If you take for $C(x)$ one particular antiderivative, then you get one particular solution $y_p(x)$, the general solution is then $y = y_p + y_h$.
If you include "+C" when deriving $C(x)$, then after substituting it into $y(x) = C(x)u(x)$ you get the general solution.

Theorem. (on **structure of solution set** of linear ODE of order 1)
Let y_p be some particular solution of the equation $y' + a(x)y = b(x)$ on an open interval I.
A function $y_0(x)$ is a solution of this equation on I if and only if $y_0 = y_p + y_h$, where $y_h(x)$ is some solution of the associated homogeneous equation on I.

Definition.
A differential equation is called **autonomous** if the free variable does not appear in it, that is, if it can be written as $F(y, y', \dots, y^{(n)}) = 0$.

Definition.
Consider a first order autonomous ODE $y' = h(y)$. $(*)$
A number y_0 is called an **equilibrium** of this equation if the constant function $y(x) = y_0$ solves $(*)$.
Such an equilibrium is called (asymptotically) **stable** if there is $d > 0$ such that for every solution $y(x)$ of the equation $(*)$ the following is true:

If $|y(x_0) - y_0| < d$ for some $x_0 \in \mathbb{R}$, then $y(x) \to y_0$ as $x \to \infty$.

Otherwise we call this equilibrium **unstable**.

Theorem. (Peano's thm on **existence**)
Consider an ODE of the form $y' = f(x,y)$. $(*)$
Let I, J be open intervals such that f is continuous on the set $I \times J$.
Then for all $(x_0, y_0) \in I \times J$ there exists a solution of the IVP $(*)$, $y(x_0) = y_0$ on some neighborhood of x_0.

Theorem. (Picard's thm on **existence and uniqueness**)
Consider an ODE of the form $y' = f(x,y)$. $(*)$
Let I, J be open intervals such that f is continuous on the set $I \times J$ and there exists $K > 0$ such that for all $x \in I$, f is K-Lipschitz as a function of y on J.
Then for all $(x_0, y_0) \in I \times J$ there exists a solution of the IVP $(*)$, $y(x_0) = y_0$ on some neighborhood of x_0 and this solution is unique on this neighborhood.

Corollary.
Consider an ODE of the form $y' = f(x,y)$. $(*)$
If I, J are open intervals such that f is continuous and $\frac{\partial f}{\partial y}$ exists and is bounded on the set $I \times J$, the through every point $(x_0, y_0) \in I \times J$ there passes exactly one solution of the equation $(*)$ and it can be extended to the boundary of $I \times J$.