14 Lecture #10: Wednesday, March 18th, 2026

14.1 Complex characteristic numbers

Consider the differential equation $y''' + y'' + y' + y = 0$. Its characteristic polynomial is $p(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1$. Factoring, we obtain

$$p(\lambda) = (\lambda + 1)(\lambda^2 + 1).$$

Hence the characteristic numbers are $\lambda = -1$ and $\lambda = \pm i$. Notice that two of these eigenvalues are complex numbers. This should imply that the expected general solution is given by

$$y(x) = C_1 e^{-x} + C_2 e^{ix} + C_3 e^{-ix}$$

for every $x \in \mathbb{R}$, where C_1, C_2, C_3 are arbitrary real numbers. This situation raises an important question: how should we interpret solutions of the form e^{ix} or e^{-ix}, since we are usually interested in real-valued solutions of differential equations? At first glance, the appearance of complex exponentials may seem problematic. However, it turns out that these complex expressions can be rewritten in terms of real-valued functions using Euler's formula. In this way, complex eigenvalues still lead to real solutions, typically involving sine and cosine functions. We need first the following fact.

Fact 14.1. Consider a linear ODE with real constant coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = b(x),$$

where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ and $b : I \to \mathbb{R}$. Suppose that a complex–valued function y is a solution of this equation. Write $y(x) = u(x) + i\, v(x)$, where u and v are the real and imaginary parts of y, respectively. Then, u is a solution of the same equation and v is a solution of the associated homogeneous equation.

Proof. Let L denote the differential operator $L[y] = y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y$. Since differentiation is linear and the coefficients a_k are real, we have $L[u + iv] = L[u] + i\, L[v]$. Because y is a solution, $L[y] = b$, hence $L[u] + i\, L[v] = b$. Since b is real–valued, this equality implies that the real and imaginary parts must match, so $L[u] = b$ and $L[v] = 0$. Therefore u solves the given (nonhomogeneous) equation, and v solves the associated homogeneous equation. $\qquad\square$

Now, suppose that $\lambda = \alpha + i\beta \in \mathbb{C}$, where $\alpha, \beta \in \mathbb{R}$. Then

$$e^{\lambda x} = e^{(\alpha + i\beta)x} = e^{\alpha x}\, e^{i\beta x}.$$

Using Euler's formula,

$$e^{i\beta x} = \cos(\beta x) + i \sin(\beta x).$$

Therefore,

$$e^{\lambda x} = e^{\alpha x}\big(\cos(\beta x) + i\sin(\beta x)\big) = e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x).$$

By the previous fact, if $e^{\lambda x}$ is a solution of a linear ODE with real coefficients, then its real and imaginary parts are also solutions (in the homogeneous case, both are solutions). Hence, from the single complex solution $e^{\lambda x}$ we obtain the two real solutions

$$e^{\alpha x}\cos(\beta x) \qquad \text{and} \qquad e^{\alpha x}\sin(\beta x).$$

Moreover, since the coefficients of the ODE are real, complex conjugation preserves solutions: if $e^{\lambda x}$ is a solution, then $e^{\overline{\lambda} x}$ is also a solution. Writing $\overline{\lambda} = \alpha - i\beta$ gives

$$e^{\overline{\lambda} x} = e^{\alpha x}\big(\cos(\beta x) - i\sin(\beta x)\big),$$

so the same two real functions appear again. Finally, the functions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ are linearly independent (for $\beta \neq 0$), so they provide two independent real solutions. We have the following result.

Example 14.2. Consider once again the differential equation $y''' + y'' + y' + y = 0$. As we have seen before its characteristic polynomial is $p(\lambda) = \lambda^3 + \lambda^2 + \lambda + 1 = (\lambda + 1)(\lambda^2 + 1)$. This gives the characteristic numbers $\lambda = -1$ and $\lambda = \pm i$. Therefore, the general solution is given by

$$y(x) = C_1 e^{-x} + C_2 e^{ix} + C_3 e^{-ix}$$

for every $x \in \mathbb{R}$. Now, as $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$, we may replace the fundamental system $\{e^{-x}, e^{ix}, e^{-ix}\}$ by the real, linearly independent pair $\{e^{-x}, \cos(x), \sin(x)\}$. Consequently, the real general solution can be written as

$$y(x) = C_1 e^{-x} + C_2 \cos(x) + C_3 \sin(x)$$

for every $x \in \mathbb{R}$ where $C_1, C_2, C_3 \in \mathbb{R}$ are arbitrary constants.

In general, we have the following result.

Theorem 14.3 (on fundamental system for LODE)**.** Consider a homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Let λ be its characteristic number of multiplicity m.

(1) If $\lambda = \alpha \in \mathbb{R}$, then

$$e^{\alpha x}, x e^{\alpha x}, \ldots, x^{m-1} e^{\alpha x}$$

are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.

(2) If $\lambda = \alpha \pm \beta i \in \mathbb{C}$, $\beta \neq 0$, then

$$e^{\alpha x}\cos(\beta x),\ x e^{\alpha x}\cos(\beta x),\ \ldots,\ x^{m-1} e^{\alpha x}\cos(\beta x),$$

$$e^{\alpha x}\sin(\beta x),\ x e^{\alpha x}\sin(\beta x),\ \ldots,\ x^{m-1} e^{\alpha x}\sin(\beta x)$$

are solutions of the associated homogeneous equation on $\mathbb{R}$ and they are linearly independent.

(3) The set of functions from (1) and (2) for all characteristic numbers is linearly independent and forms a fundamental system of the given equation on $\mathbb{R}$.

14.2   Asymptotic rate of growth at infinity of solutions

In many applications, differential equations are used to model the evolution of a system over time. In such situations, it is often important not only to find the exact solutions of the equation, but also to understand how these solutions behave when time becomes very large. In other words, we are interested in the *asymptotic behavior* of the solutions as $x \to +\infty$. This type of analysis allows us to answer questions

such as: does the solution grow without bound, does it decay to zero, or does it remain bounded and oscillate? Recall we have done something in this direction in Problem 3.2.

The *asymptotic rate of growth* describes how fast a solution increases or decreases as x becomes large. For example, exponential terms like e^{ax} grow very quickly when $a > 0$ and decay rapidly when $a < 0$. On the other hand, trigonometric functions such as $\cos(x)$ and $\sin(x)$ remain bounded for all x, so they produce oscillatory behavior rather than growth or decay. By identifying which terms dominate a solution for large values of x, we can obtain a clear qualitative understanding of the long-term behavior of the system being modeled.

We now discuss the asymptotic behavior of the following solutions as $x \to +\infty$. The main idea is simple: among terms such as e^{ax}, the one with the *largest* exponent a dominates the growth as $x \to +\infty$. In particular,

   ⋆ if $a > 0$, then $e^{ax} \to +\infty$ (exponential growth),

   ⋆ if $a = 0$, then $e^{ax} = 1$ (constant size) and

   ⋆ if $a < 0$, then $e^{ax} \to 0$ (exponential decay).

Moreover, polynomial factors such as x do not change the fact that an exponential term dominates; for instance, xe^x still grows exponentially and it is even larger than e^x when x goes to ∞.

1) $y(x) = 1 - e^x + xe^x + 3e^{-2x}$. As $x \to +\infty$, we have $e^{-2x} \to 0$, so the term $3e^{-2x}$ becomes negligible. The constant 1 is also negligible compared with e^x-terms. Thus, the behavior is determined by

$$-e^x + xe^x = (x-1)e^x.$$

Since $(x-1)e^x \to +\infty$, we conclude that

$$y(x) \sim (x-1)e^x \quad \text{as } x \to +\infty,$$

and in particular $y(x) \to +\infty$ with exponential rate like xe^x.

2) $y(x) = C_1 + C_2e^x + C_3xe^x + C_4e^{-2x}$. As $x \to +\infty$, we have $e^{-2x} \to 0$, so the term C_4e^{-2x} decays to 0. The constant term C_1 does not grow. The dominant terms are C_2e^x and C_3xe^x. Notice that xe^x grows faster than e^x, so:

   ⋆ if $C_3 \neq 0$, then the dominant term is C_3xe^x and
$$y(x) \sim C_3xe^x \quad \text{as } x \to +\infty,$$
   so $y(x) \to +\infty$ if $C_3 > 0$ and $y(x) \to -\infty$ if $C_3 < 0$;

   ⋆ if $C_3 = 0$ but $C_2 \neq 0$, then
$$y(x) \sim C_2e^x \quad \text{as } x \to +\infty,$$
   so $y(x) \to +\infty$ if $C_2 > 0$ and $y(x) \to -\infty$ if $C_2 < 0$;

   ⋆ if $C_2 = C_3 = 0$, then $y(x) \to C_1$ (since $C_4e^{-2x} \to 0$).

3) $y(x) = C_1e^x + C_2e^{2x}$. Here e^{2x} grows much faster than e^x as $x \to +\infty$. Therefore:

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- if $C_2 \neq 0$, then
$$y(x) \sim C_2 e^{2x} \quad \text{as } x \to +\infty,$$
so $y(x) \to +\infty$ if $C_2 > 0$ and $y(x) \to -\infty$ if $C_2 < 0$;

- if $C_2 = 0$, then $y(x) = C_1 e^x$, so it grows like e^x (or is identically 0 if $C_1 = 0$).

4) $y(x) = C_1 e^{-x} + C_2 \cos(x) + C_3 \sin(x)$. As $x \to +\infty$, we have $e^{-x} \to 0$, so the term $C_1 e^{-x}$ vanishes. The remaining terms are sinusoidal. Since $\cos(x)$ and $\sin(x)$ are bounded between -1 and 1, any linear combination
$$C_2 \cos(x) + C_3 \sin(x)$$
remains bounded for all x. In fact, it can be rewritten in amplitude–phase form:

$$C_2 \cos(x) + C_3 \sin(x) = R \cos(x - \delta), \qquad R = \sqrt{C_2^2 + C_3^2}.$$

Thus, as $x \to +\infty$ the solution does *not* grow; instead it oscillates between $-R$ and R, and we cannot provide any answer in this case. In particular, $y(x) \to 0$ if and only if $C_2 = C_3 = 0$ (because then $y(x) = C_1 e^{-x} \to 0$).

5) $y(x) = C_1 e^x + C_2 \cos(x) + C_3 \sin(x)$. Here the exponential term $C_1 e^x$ dominates the oscillatory terms as $x \to +\infty$, because $\cos(x)$ and $\sin(x)$ are bounded while $e^x \to +\infty$. Therefore:

- if $C_1 \neq 0$, then
$$y(x) \sim C_1 e^x \quad \text{as } x \to +\infty,$$
so $y(x) \to +\infty$ if $C_1 > 0$ and $y(x) \to -\infty$ if $C_1 < 0$;

- if $C_1 = 0$, then $y(x) = C_2 \cos(x) + C_3 \sin(x)$, which is bounded and oscillatory (as in item 4).

14.3   An example of a homogeneous equation with a parameter

In this short section, we consider the differential equation

$$y'' - 6y' + 9py = 0 \tag{21}$$

where $p \in \mathbb{R}$ is a parameter and analyze it according to it. Situations of this type appear frequently in applications, since the parameter p often represents a physical quantity (for instance a damping coefficient, a reaction rate, or a feedback parameter). Changing the value of p may change the qualitative behavior of the solutions, so it is important to understand how the solutions depend on this parameter.

Knowing for which values of p the set of solutions does *not* contain both a positive increasing solution and a positive decreasing solution is important because it gives information about the possible long–term behaviors of the system. If both types of solutions exist, then the system may either grow or decay depending on the initial conditions, which indicates that the model allows qualitatively different outcomes. On the other hand, if such a pair of solutions cannot occur for certain values of p, then the behavior of the system becomes more predictable: all solutions may grow, all may decay, or they may oscillate without exhibiting both monotone growth and decay. Understanding these regimes helps us determine how the parameter p influences the stability and qualitative dynamics of the system.

Therefore, here, we would like to know for which values of p it is guaranteed that the set of solutions for the equation (21) does *not* include both a positive increasing solution and a positive decreasing solution.

We look for solutions of the form $y = e^{\lambda x}$. Notice that what we do not want is that we find at the same time solutions of the form $ae^{\lambda_1 x}$ and be^{λ_2} where $a, b > 0$, $\lambda_1 > 0$ and $\lambda_2 < 0$. The characteristic polynomial is $p(\lambda) = \lambda^2 - 6\lambda + 9p$. so the characteristic equation is $\lambda^2 - 6\lambda + 9p = 0$. Solving, we obtain

$$\lambda = \frac{6 \pm \sqrt{36 - 36p}}{2} = 3 \pm 3\sqrt{1 - p}.$$

We now analyze the different cases.

⋆ If $p < 0$, then $\sqrt{1 - p} > 1$, so

$$\lambda_1 = 3 + 3\sqrt{1 - p} > 0 \quad \text{and} \quad \lambda_2 = 3 - 3\sqrt{1 - p} < 0.$$

Hence the solution set contains both $e^{\lambda_1 x}$ (which is positive and increasing) and $e^{\lambda_2 x}$ (which is positive and decreasing). This is exactly what we want to prevent.

⋆ If $p = 0$, then the roots are $\lambda_1 = 6$ and $\lambda_2 = 0$. The corresponding solutions are e^{6x} and 1. Thus we have a positive increasing solution and a constant solution, but no positive decreasing solution.

⋆ If $0 < p < 1$, then $0 < \sqrt{1 - p} < 1$, so

$$\lambda_1 = 3 + 3\sqrt{1 - p} > 0 \quad \text{and} \quad \lambda_2 = 3 - 3\sqrt{1 - p} > 0.$$

In this case both exponential solutions grow, so there is no positive decreasing solution.

⋆ If $p = 1$, then the characteristic root is $\lambda = 3$ with multiplicity 2, and the general solution is

$$y(x) = C_1 e^{3x} + C_2 x e^{3x} = (C_1 + C_2 x)e^{3x}.$$

Again, there is no positive decreasing solution.

⋆ If $p > 1$, then the characteristic roots are complex $\lambda = 3 \pm 3i\sqrt{p - 1}$. The solutions are of the form

$$y(x) = C_1 \cdot e^{3x} \cdot \cos(3\sqrt{p - 1}\, x) + C_2 \cdot e^{3x} \cdot \sin(3\sqrt{p - 1}\, x))$$

which grow exponentially in magnitude. Hence there is no positive decreasing solution.

Now we can safely conclude that the only case in which the solution set contains both a positive increasing solution and a positive decreasing solution is when $p < 0$. Consequently, the values of p for which this cannot happen are $p \geq 0$.