

16 Lecture #11: Tuesday, March 24th, 2026

16.1 Structural theorems for non-homogeneous linear ODEs

From Theorem 13.5, in order to solve a non-homogeneous linear ordinary differential equation, it suffices to determine a particular solution. Indeed, every solution of such an equation can be written in the form $y = y_p + y_h$, where y_p is a particular solution of the nonhomogeneous equation and y_h denotes the general solution of the associated homogeneous equation, which we already know how to find. For the convenience of the reader and to avoid that they are jumping into different PDFs, we restate Theorem 13.5 here.

Theorem 16.1. Let y_p be some particular solution of a given linear ODE on an open interval I . A function y_0 is a solution of this equation on I if and only if

$$y_0 = y_p + y_h$$

for some solution y_h of the associated homogeneous equation on I .

In what follows, we illustrate the idea of the **method of undetermined coefficients** (see Theorem 16.2 below) through a few examples. The goal is to provide a method where we will be able to find a particular solution y_p of a non-homogeneous equation. From the theorem above, we will be able to give the general solution of a linear ODE with constant coefficients.

16.2 The method of undetermined coefficients for special RHS

The key idea in this section is to *guess* the form of the particular solution based on the right-hand side $b(x)$ (here, we write RHS, for short), leaving some coefficients undetermined. These coefficients are then found by substituting the guess into the differential equation. As we believe that the most efficient way to learn this method is through examples, in what follows, we will consider first linear ODEs of the form

$$y' - 2y = b(x)$$

where $b(x)$ is considered to be $b(x) = e^x$, $b(x) = x$ or $b(x) = \cos(x)$. Consider first $y' - 2y = e^x$. Since the RHS is an exponential function, we try a particular solution of the same form

$$y_p = Ae^x$$

where A is an unknown constant that we should determine. In order to find A , we compute $y_p' = Ae^x$. Substituting into the differential equation gives $Ae^x - 2Ae^x = e^x$. Hence $(-A)e^x = e^x$, which implies $A = -1$. Therefore a particular solution is $y_p = -e^x$.

Now consider $y' - 2y = x$. Since the RHS is a polynomial of degree 1, we try a polynomial of the same degree, that is,

$$y_p = Ax + B$$

where A and B are constants to be determined. Once again, we compute $y_p' = A$. Substituting into the equation gives $A - 2(Ax + B) = x$. Expanding, $A - 2Ax - 2B = x$. Rearranging terms, $-2Ax + (A - 2B) = x$. Matching the coefficients of equal powers of x , we obtain $-2A = 1$ and $A - 2B = 0$. Thus, $A = -\frac{1}{2}$ and $B = -\frac{1}{4}$. A particular solution is therefore given by

$$y_p = -\frac{1}{2}x - \frac{1}{4}.$$

Now consider the equation $y' - 2y = \cos x$. For a cosine RHS, we try a combination of sine and cosine, that is, we need to find A and B such that

$$y_p = A \cos x + B \sin x.$$

We compute $y'_p = -A \sin x + B \cos x$. Substituting into the differential equation,

$$(-A \sin x + B \cos x) - 2(A \cos x + B \sin x) = \cos x.$$

Expanding,

$$(B - 2A) \cos x + (-A - 2B) \sin x = \cos x.$$

Matching coefficients gives $B - 2A = 1$ and $-A - 2B = 0$. From the second equation, $A = -2B$. Substituting into the first equation, $B - 2(-2B) = 1$, so $5B = 1$, that is, $B = \frac{1}{5}$. Thus $A = -\frac{2}{5}$. A particular solution is therefore given by

$$y_p = -\frac{2}{5} \cos x + \frac{1}{5} \sin x.$$

In general, we have the following result which teaches us how to find a possible particular solution for a non-homogeneous linear differential equation with constant coefficients.

Theorem 16.2 (On guessing a solution for special right hand side). Consider a linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = b(x).$$

Assume that

$$b(x) = e^{\alpha x} [P(x) \cos(\beta x) + Q(x) \sin(\beta x)]$$

for some polynomials P, Q , and denote

$$d = \max(\deg(P), \deg(Q)).$$

Let m be the multiplicity of the number $\lambda = \alpha \pm \beta i$ as a characteristic number of the associated homogeneous equation (we put $m = 0$ if it is not a characteristic number at all). Then there exist polynomials \tilde{P}, \tilde{Q} of degree at most d such that

$$y(x) = x^m e^{\alpha x} [\tilde{P}(x) \cos(\beta x) + \tilde{Q}(x) \sin(\beta x)]$$

is a solution of the given equation on \mathbb{R} . This is called the **method of undetermined coefficients**.

As we have mentioned before, the method of undetermined coefficients is best learned through practice. Although the general principle is simple - guessing the form of a particular solution based on the RHS and determining the unknown coefficients by substitution - it becomes intuitive only after working through many examples. In practice, one must become familiar with which type of guess corresponds to each type of RHS (here, we will be working with exponentials, polynomials, cosines and sinus, or products and combinations of these). The only effective way to develop this intuition is to solve a variety of exercises. For this reason, the reader is encouraged to practice with many different examples until the choice of the appropriate *ansatz*^{††} becomes natural.

In order to sum up this approach, simpler forms of incomplete RHS are given in the what follows. Suppose that $\lambda = \alpha \pm \beta i$ is a solution of the homogeneous polynomial and we put $m = 0$ if λ is not a characteristic number.

^{††}*Ansatz* is a term used in Mathematics and Physics. It comes from German and literally means “approach”, “initial placement” or “starting assumption”.

- ★ If $b(x) = P(x)$, then $y(x) = x^m \tilde{P}(x)$, where m is the multiplicity of $\lambda = 0$.
- ★ If $b(x) = P(x)e^{\alpha x}$, then $y(x) = x^m \tilde{P}(x)e^{\alpha x}$, where m is the multiplicity of $\lambda = \alpha$.
- ★ If $b(x) = P(x) \cos(\beta x) + Q(x) \sin(\beta x)$, then $y(x) = x^m [\tilde{P}(x) \cos(\beta x) + \tilde{Q}(x) \sin(\beta x)]$, where m is the multiplicity of $\lambda = \beta i$.

Let us practice it.

Example 16.3. Consider the differential equation

$$y' - 2y = 8e^{2x}.$$

We solve first the associated homogeneous equation. The homogeneous equation is $y' - 2y = 0$. The characterization equation is given by $\lambda - 2 = 0$, which means that $y_h = Ce^{2x}$ for every $x \in \mathbb{R}$. Now we need to find a particular solution. Since the RHS is $8e^{2x}$ and $\lambda = 2$ is already a solution of the homogeneous polynomial, we multiply our usual guess by x and try $y_p = Axe^{2x}$. Then, $y_p' = Ae^{2x} + 2Axe^{2x}$. Now we substitute into the equation to get

$$(Ae^{2x} + 2Axe^{2x}) - 2(Axe^{2x}) = 8e^{2x}.$$

The $2Axe^{2x}$ terms cancel, leaving $Ae^{2x} = 8e^{2x}$, which it turn implies that $A = 8$. Thus, $y_p = 8xe^{2x}$. Therefore, the general solution is given by

$$y(x) = y_h + y_p = Ce^{2x} + 8xe^{2x}$$

for every $x \in \mathbb{R}$.

16.3 The superposition principle

The superposition principle allows us to solve more complicated non-homogeneous equations by decomposing the RHS into simpler parts, finding a particular solution for each part, and then adding these solutions together. Indeed, we have the following theorem.

Theorem 16.4 (Superposition principle). Consider a linear ODE with left hand-side

$$L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y.$$

Let y_1 be a solution of $L(y) = b_1(x)$ on an open interval I and y_2 be a solution of $L(y) = b_2(x)$ on I . Then $y_1 + y_2$ is a solution of

$$L(y) = b_1(x) + b_2(x)$$

on I .

Now we will further apply Theorems 16.2 and 16.4. Note below that examples of this length are not expected on exams. Nevertheless, studying them helps the reader develop a complete understanding of the methods discussed here for solving linear ODEs with constant coefficients.

Example 16.5. We solve

$$y' + 2y = 9e^{-x} \sin(2x) + 5xe^{-x} \cos(2x)$$

using the characteristic polynomial for the homogeneous equation, the method of undetermined coefficients for particular solutions and finally we apply the superposition principle. We split the solution of this example into small steps.

★ *Step 1:* Consider the homogeneous equation $y' + 2y = 0$. Its characteristic polynomial is $\lambda + 2 = 0$, so $\lambda = -2$ and therefore $y_h = Ce^{-2x}$ ($x \in \mathbb{R}$) is the general solution of the associated homogeneous equation.

★ *Step 2:* Let us write the RHS as a sum of the form $b(x) = b_1(x) + b_2(x)$, where $b_1(x) = 9e^{-x} \sin(2x)$ and $b_2(x) = 5xe^{-x} \cos(2x)$. By the superposition principle, if $y_{p,1}$ solves $y' + 2y = b_1(x)$ and $y_{p,2}$ solves $y' + 2y = b_2(x)$, then $y_p = y_{p,1} + y_{p,2}$ solves $y' + 2y = b(x)$.

★ *Step 3:* Let us find now a particular solution for the RHS $b_1(x) = 9e^{-x} \sin(2x)$. Since $b_1(x)$ has the form $e^{\alpha x} \sin(\beta x)$ with $\alpha = -1$ and $\beta = 2$, we try to find a particular solution as

$$y_{p,1} = x^0 \cdot e^{-x} (A \cos(2x) + B \sin(2x)) = e^{-x} (A \cos(2x) + B \sin(2x))$$

(notice that $m = 0$ as $\lambda = \alpha \pm i\beta = -1 \pm 2i$ is not a characteristic number). Compute

$$y'_{p,1} = e^{-x} \left((-A + 2B) \cos(2x) + (-B - 2A) \sin(2x) \right),$$

hence

$$y'_{p,1} + 2y_{p,1} = e^{-x} \left((A + 2B) \cos(2x) + (B - 2A) \sin(2x) \right).$$

We match coefficients with $9e^{-x} \sin(2x)$ and obtain the system

$$\begin{cases} A + 2B = 0, \\ -2A + B = 9. \end{cases}$$

Solving gives $A = -\frac{18}{5}$, $B = \frac{9}{5}$, so

$$y_{p,1} = e^{-x} \left(-\frac{18}{5} \cos(2x) + \frac{9}{5} \sin(2x) \right).$$

★ *Step 4:* Let us find a particular for $b_2(x) = 5xe^{-x} \cos(2x)$. Now the RHS is a polynomial of degree 1 times $e^{-x} \cos(2x)$, so we try the following particular solution

$$y_{p,2} = e^{-x} \left((Ax + B) \cos(2x) + (Cx + D) \sin(2x) \right).$$

Let $u(x) = (Ax + B) \cos(2x) + (Cx + D) \sin(2x)$, so $y_{p,2}(x) = e^{-x}u(x)$. Then

$$y'_{p,2} + 2y_{p,2} = e^{-x}(u' + u).$$

Compute

$$u' = (A + 2Cx + 2D) \cos(2x) + (C - 2Ax - 2B) \sin(2x),$$

so

$$u' + u = \left((A + 2Cx + 2D) + (Ax + B) \right) \cos(2x) + \left((C - 2Ax - 2B) + (Cx + D) \right) \sin(2x).$$

Thus

$$y'_{p,2} + 2y_{p,2} = e^{-x} \left(\left((2C + A)x + (A + 2D + B) \right) \cos(2x) + \left((-2A + C)x + (C - 2B + D) \right) \sin(2x) \right).$$

We match this with $5xe^{-x} \cos(2x)$, i.e., with $e^{-x} (5x \cos(2x) + 0 \cdot \cos(2x) + 0 \cdot \sin(2x))$. Hence the coefficient equations yield the following system

$$\begin{cases} 2C + A = 5, \\ A + 2D + B = 0, \\ -2A + C = 0, \\ C - 2B + D = 0. \end{cases}$$

From $-2A + C = 0$ we get $C = 2A$. Substituting into $2C + A = 5$ gives $4A + A = 5$, so $A = 1$ and $C = 2$. Then $C - 2B + D = 0$ gives $2 - 2B + D = 0$, i.e. $D = 2B - 2$. Finally $A + 2D + B = 0$ becomes $1 + 2(2B - 2) + B = 0$, so $1 + 4B - 4 + B = 0$, hence $5B = 3$ and

$$B = \frac{3}{5} \quad \text{and} \quad D = 2 \cdot \frac{3}{5} - 2 = -\frac{4}{5}.$$

Therefore,

$$y_{p,2} = e^{-x} \left(\left(x + \frac{3}{5} \right) \cos(2x) + \left(2x - \frac{4}{5} \right) \sin(2x) \right).$$

★ *Step 5:* We finally combine these particular solutions by superposition and add the homogeneous part for the general solution. By superposition, we have that $y_p = y_{p,1} + y_{p,2}$. Thus the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= Ce^{-2x} + e^{-x} \left(-\frac{18}{5} \cos(2x) + \frac{9}{5} \sin(2x) \right) + e^{-x} \left(\left(x + \frac{3}{5} \right) \cos(2x) + \left(2x - \frac{4}{5} \right) \sin(2x) \right) \\ &= Ce^{-2x} + e^{-x} \left((x - 3) \cos(2x) + (2x + 1) \sin(2x) \right) \end{aligned}$$

for every $x \in \mathbb{R}$ and for an arbitrary constant $C \in \mathbb{R}$.

Remark 16.6. Notice that we do not need to use the superposition principle in Example 16.5. Indeed, we could have adventured ourselves and guessed a particular solution directly by using the method of undetermined coefficients. Notice that the RHS is

$$9e^{-x} \sin(2x) + 5xe^{-x} \cos(2x) = e^{-x} (9 \sin(2x) + 5x \cos(2x)).$$

Thus it is an exponential factor e^{-x} multiplied by a polynomial of degree 1 in x times $\sin(2x)$ or $\cos(2x)$. According to the guessing rule for the method of undetermined coefficients, if the forcing term has the form $e^{\alpha x} (P_n(x) \cos(\beta x) + Q_n(x) \sin(\beta x))$, where P_n and Q_n are polynomials of degree n , then we try a particular solution of the form

$$y_p = e^{\alpha x} (\tilde{P}_n(x) \cos(\beta x) + \tilde{Q}_n(x) \sin(\beta x))$$

where \tilde{P}_n and \tilde{Q}_n are polynomials of the same degree with undetermined coefficients. In our case $\alpha = -1$ and $\beta = 2$, and the polynomial degree is 1. Therefore we take polynomials of degree 1, that is, $\tilde{P}_1(x) = Ax + B$ and $\tilde{Q}_1(x) = Cx + D$, which leads to the guess

$$y_p = e^{-x} \left((Ax + B) \cos(2x) + (Cx + D) \sin(2x) \right).$$

Finally we verify that $-1 \pm 2i$ is not a root of the characteristic polynomial $\lambda + 2 = 0$. Hence no extra power of x is required in the guess above.

Example 16.7. We solve

$$y'' - 2y' + y = e^x + 27e^{-2x} + 13 - 25\sin(2x)$$

using the characteristic polynomial for the homogeneous equation, the method of undetermined coefficients for particular solutions and finally we apply the superposition principle. We split the solution of this example into small steps. Different from Example 16.5 here we are forced to use superposition as there is no way to group the terms of the RHS together.

★ *Step 1:* Consider the homogeneous equation $y'' - 2y' + y = 0$. Its characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, so $\lambda = 1$ is a root of multiplicity 2. Therefore, $y_h = C_1e^x + C_2xe^x$ ($x \in \mathbb{R}$) is the general solution of the associated homogeneous equation.

★ *Step 2:* Let us write the RHS as a sum of the form $b(x) = b_1(x) + b_2(x) + b_3(x) + b_4(x)$, where $b_1(x) = e^x$, $b_2(x) = 27e^{-2x}$, $b_3(x) = 13$ and $b_4(x) = -25\sin(2x)$. By the superposition principle, if $y_{p,1}$ solves $y'' - 2y' + y = b_1(x)$, $y_{p,2}$ solves $y'' - 2y' + y = b_2(x)$, $y_{p,3}$ solves $y'' - 2y' + y = b_3(x)$ and $y_{p,4}$ solves $y'' - 2y' + y = b_4(x)$, then

$$y_p = y_{p,1} + y_{p,2} + y_{p,3} + y_{p,4}$$

solves the original equation $y'' - 2y' + y = b(x)$, which is what we want.

★ *Step 3:* Let us find a particular solution for the RHS $b_1(x) = e^x$. Since e^x corresponds to the root $\lambda = 1$ of multiplicity 2 of the characteristic polynomial, we multiply the usual guess by x^2 and try

$$y_{p,1} = Ax^2e^x.$$

Compute

$$y'_{p,1} = Ae^x(x^2 + 2x) \quad \text{and} \quad y''_{p,1} = Ae^x(x^2 + 4x + 2).$$

Hence

$$y''_{p,1} - 2y'_{p,1} + y_{p,1} = Ae^x[(x^2 + 4x + 2) - 2(x^2 + 2x) + x^2] = 2Ae^x.$$

We match coefficients with e^x and obtain $2A = 1$, hence $A = \frac{1}{2}$. Therefore, we have that

$$y_{p,1} = \frac{1}{2}x^2e^x.$$

★ *Step 4:* Let us find a particular solution for $b_2(x) = 27e^{-2x}$. Since the RHS has the form $e^{\alpha x}$ with $\alpha = -2$ and -2 is not a characteristic number, we try

$$y_{p,2} = Be^{-2x}.$$

Compute

$$y'_{p,2} = -2Be^{-2x} \quad \text{and} \quad y''_{p,2} = 4Be^{-2x}.$$

Hence

$$y''_{p,2} - 2y'_{p,2} + y_{p,2} = (4B - 2(-2B) + B)e^{-2x} = 9Be^{-2x}.$$

Matching coefficients with $27e^{-2x}$ gives $9B = 27$, so $B = 3$ and therefore

$$y_{p,2} = 3e^{-2x}.$$

★ *Step 5:* Let us find a particular solution for $b_3(x) = 13$. Since the RHS is a constant, we try

$$y_{p,3} = K.$$

Then $y'_{p,3} = 0$ and $y''_{p,3} = 0$, so $y''_{p,3} - 2y'_{p,3} + y_{p,3} = K$. Matching with 13 gives $K = 13$. Therefore, we have that

$$y_{p,3} = 13.$$

★ *Step 6:* Let us find a particular solution for $b_4(x) = -25 \sin(2x)$. Since the RHS has the form $\sin(\beta x)$ with $\beta = 2$, we try

$$y_{p,4} = A \cos(2x) + B \sin(2x).$$

Compute

$$y'_{p,4} = -2A \sin(2x) + 2B \cos(2x) \quad \text{and} \quad y''_{p,4} = -4A \cos(2x) - 4B \sin(2x).$$

Hence

$$y''_{p,4} - 2y'_{p,4} + y_{p,4} = (-3A - 4B) \cos(2x) + (4A - 3B) \sin(2x).$$

We match coefficients with $-25 \sin(2x)$ and obtain the system

$$\begin{cases} -3A - 4B = 0, \\ 4A - 3B = -25. \end{cases}$$

From the first equation $A = -\frac{4}{3}B$. Substituting into the second equation gives

$$4 \left(-\frac{4}{3}B \right) - 3B = -25$$

which yields $B = 3$ and $A = -4$. Hence

$$y_{p,4} = -4 \cos(2x) + 3 \sin(2x).$$

★ *Step 7:* We finally combine these particular solutions by superposition and add the homogeneous part for the general solution. By superposition, we have $y_p = y_{p,1} + y_{p,2} + y_{p,3} + y_{p,4}$. Thus the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x + 3e^{-2x} + 13 - 4 \cos(2x) + 3 \sin(2x) \end{aligned}$$

for every $x \in \mathbb{R}$ and for arbitrary constants $C_1, C_2 \in \mathbb{R}$.