

17 Lecture #12: Wednesday, March 25th, 2026

17.1 Method of variation for linear ODEs

In this section we introduce another method for finding a particular solution of a non-homogeneous differential equation. For motivation, let us go back to the first-order equation $y' + a(x)y = b(x)$. Recall how we solved this equation. First we considered the homogeneous equation $y' + a(x)y = 0$, which can be solved by separation of variables. Its general solution can be written in the form $y_h(x) = C u(x)$, where C is an arbitrary constant and $u(x)$ is an exponential function that solves the homogeneous equation. The key idea of the variation method is to replace the constant C by a function $C(x)$. In this way we look for a particular solution of the form $y_p(x) = C(x)u(x)$. This idea is somewhat reminiscent of the guessing method: we propose a form for the particular solution and determine the unknown function from the equation. To determine $C(x)$ we substitute $y_p(x) = C(x)u(x)$ into the differential equation. First compute $y_p'(x) = C'(x)u(x) + C(x)u'(x)$. Substituting into the equation gives $C'(x)u(x) + C(x)u'(x) + a(x)C(x)u(x) = b(x)$. Rearranging the terms we obtain

$$C'(x)u(x) + C(x)(u'(x) + a(x)u(x)) = b(x).$$

Since $u(x)$ solves the homogeneous equation, we have $u'(x) + a(x)u(x) = 0$. Therefore the second term vanishes and we obtain $C'(x)u(x) = b(x)$. From this equation we can determine $C'(x)$ and then find $C(x)$ by integration. The idea of the method of variation of parameters is to extend this same strategy to second-order linear differential equations.

Consider the equation

$$y'' + a_1(x)y' + a_0(x)y = b(x).$$

First we consider the homogeneous equation

$$y'' + a_1(x)y' + a_0(x)y = 0.$$

Assume that we can find a fundamental system of solutions for this equation. Then every solution of the homogeneous equation can be written as

$$y_h(x) = C_1 u_1(x) + C_2 u_2(x),$$

where C_1 and C_2 are arbitrary constants.

Now we look for a particular solution of the form

$$y_p(x) = C_1(x)u_1(x) + C_2(x)u_2(x).$$

We substitute this guess into the equation and hope that some terms cancel out. On the one hand, we have

$$y_p' = C_1'(x)u_1(x) + C_2'(x)u_2(x) + C_1(x)u_1'(x) + C_2(x)u_2'(x),$$

and

$$y_p'' = C_1''(x)u_1(x) + C_2''(x)u_2(x) + 2C_1'(x)u_1'(x) + 2C_2'(x)u_2'(x) + C_1(x)u_1''(x) + C_2(x)u_2''(x).$$

If we substitute the expressions for y_p' and y_p'' directly into the equation, we obtain a very long expression containing the terms $C_1''(x)$, $C_2''(x)$, $C_1'(x)$, $C_2'(x)$, and also the functions u_1 , u_2 and their derivatives. Although some terms involving u_1'' and u_2'' can be simplified because u_1 and u_2 solve the homogeneous

equation, the resulting expression is still quite complicated and does not immediately allow us to determine $C_1(x)$ and $C_2(x)$.

For this reason, instead of working with the full expression we introduce additional constraints on $C_1(x)$ and $C_2(x)$ that simplify the calculations. The idea is to impose conditions that eliminate some of the terms appearing in y_p' and y_p'' , making the substitution into the differential equation much easier.

A convenient choice is to require that

$$C_1'(x)u_1(x) + C_2'(x)u_2(x) = 0.$$

With this condition, the expression for y_p' simplifies to $y_p' = C_1(x)u_1'(x) + C_2(x)u_2'(x)$. When we differentiate once more, the expression for y_p'' becomes much simpler as well, and after substituting into the differential equation we obtain another equation involving only $C_1'(x)$ and $C_2'(x)$. Indeed, in this new case, we get

$$y_p''(x) = C_1'(x)u_1'(x) + C_2'(x)u_2'(x) + C_1(x)u_1''(x) + C_2(x)u_2''(x).$$

Now we substitute all of these equations into our original one. We stop writing “ (x) ” just to simplify the notation. As the equation is $y_p'' + a_1(x)y_p' + a_0(x)y_p = b(x)$, we have

$$(C_1'u_1' + C_2'u_2' + C_1u_1'' + C_2u_2'') + a_1(x)(C_1u_1' + C_2u_2') + a_0(x)(C_1u_1 + C_2u_2) = b(x).$$

Equivalently,

$$(C_1'u_1' + C_2'u_2') + C_1(u_1'' + a_1(x)u_1' + a_0(x)u_1) + C_2(u_2'' + a_1(x)u_2' + a_0(x)u_2) = b(x).$$

As $u_1'' + a_1(x)u_1' + a_0(x)u_1 = u_2'' + a_1(x)u_2' + a_0(x)u_2 = 0$, we have that $C_1'u_1' + C_2'u_2' = b(x)$. Therefore, we have the following system

$$\begin{cases} C_1'u_1 + C_2'u_2 = 0, \\ C_1'u_1' + C_2'u_2' = b(x). \end{cases} \quad (22)$$

These ideas are part of a much wider result.

Algorithm 17.1 (variation of parameter method for solving LODE). Given the equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = b(x).$$

1. Using characteristic numbers, find a general solution y_h of the associated homogeneous equation. It has the form

$$y_h(x) = c_1u_1(x) + \cdots + c_nu_n(x).$$

2. Variation of parameter: seek a solution of the form

$$y(x) = c_1(x)u_1(x) + \cdots + c_n(x)u_n(x).$$

The unknown functions $c_i(x)$ are found by solving the system of equations

$$\begin{cases} c_1'(x)u_1(x) + \cdots + c_n'(x)u_n(x) = 0, \\ c_1'(x)u_1'(x) + \cdots + c_n'(x)u_n'(x) = 0, \\ \vdots \\ c_1'(x)u_1^{(n-2)}(x) + \cdots + c_n'(x)u_n^{(n-2)}(x) = 0, \\ c_1'(x)u_1^{(n-1)}(x) + \cdots + c_n'(x)u_n^{(n-1)}(x) = b(x). \end{cases}$$

Solve for $c'_i(x)$, integrate them to obtain $c_i(x)$, and substitute these into

$$y(x) = \sum_{i=1}^n c_i(x)u_i(x).$$

3. If you take for each $c_i(x)$ one particular antiderivative, then you obtain one particular solution $y_p(x)$. The general solution is then

$$y = y_p + y_h.$$

If you include “+C” when integrating $c'_i(x)$, then after substituting them into

$$y(x) = \sum_{i=1}^n c_i(x)u_i(x)$$

you obtain the general solution directly.

Going back to (22). The system

$$\begin{cases} C'_1(x)u_1(x) + C'_2(x)u_2(x) = 0, \\ C'_1(x)u'_1(x) + C'_2(x)u'_2(x) = b(x) \end{cases}$$

is a linear system for the unknown functions $C'_1(x)$ and $C'_2(x)$. At each fixed x , the coefficient matrix of this system is

$$\begin{pmatrix} u_1(x) & u_2(x) \\ u'_1(x) & u'_2(x) \end{pmatrix}.$$

This system has a solution whenever this matrix is invertible, that is, whenever its determinant is nonzero. This determinant is

$$W(u_1, u_2)(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u'_1(x) & u'_2(x) \end{vmatrix} = u_1(x)u'_2(x) - u_2(x)u'_1(x),$$

which is called the Wronskian (see Definition 13.7) of u_1 and u_2 . Since u_1 and u_2 form a fundamental system of solutions of the homogeneous equation, they are linearly independent. For a second-order linear differential equation, this implies that the Wronskian is not zero on the interval under consideration. Therefore, $W(u_1, u_2)(x) \neq 0$ and the coefficient matrix is invertible. Consequently, the system has a unique solution for $C'_1(x)$ and $C'_2(x)$. In fact, using Cramer’s rule we obtain

$$C'_1(x) = \frac{\begin{vmatrix} 0 & u_2(x) \\ b(x) & u'_2(x) \end{vmatrix}}{W(u_1, u_2)(x)} = -\frac{u_2(x)b(x)}{W(u_1, u_2)(x)} \quad \text{and} \quad C'_2(x) = \frac{\begin{vmatrix} u_1(x) & 0 \\ u'_1(x) & b(x) \end{vmatrix}}{W(u_1, u_2)(x)} = \frac{u_1(x)b(x)}{W(u_1, u_2)(x)}.$$

Let us put this into practice.

Example 17.2. We solve

$$y'' - 2y' + y = e^x + 27e^{-2x} + 13$$

using the method of variation of parameters.

★ *Step 1:* Consider the homogeneous equation $y'' - 2y' + y = 0$. As we have seen before, its characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$ so $\lambda = 1$ is a root of multiplicity 2. Therefore a fundamental system of solutions is $u_1(x) = e^x$ and $u_2(x) = xe^x$ and the general solution of the homogeneous equation is

$$y_h(x) = C_1e^x + C_2xe^x.$$

★ *Step 2:* By variation of parameters we look for a particular solution of the form

$$y_p(x) = C_1(x)e^x + C_2(x)xe^x.$$

The functions $C_1(x)$ and $C_2(x)$ must satisfy the system

$$\begin{cases} C_1'(x)e^x + C_2'(x)xe^x = 0, \\ C_1'(x)e^x + C_2'(x)(x+1)e^x = b(x), \end{cases}$$

where

$$b(x) = e^x + 27e^{-2x} + 13.$$

★ *Step 3:* Subtract the first equation from the second one. We obtain

$$C_2'(x)e^x = b(x).$$

Hence

$$C_2'(x) = \frac{b(x)}{e^x} = 1 + 27e^{-3x} + 13e^{-x}.$$

Integrating gives

$$C_2(x) = \int (1 + 27e^{-3x} + 13e^{-x}) dx = x - 9e^{-3x} - 13e^{-x}.$$

★ *Step 4:* Substitute $C_2'(x)$ into the first equation

$$C_1'(x)e^x + C_2'(x)xe^x = 0.$$

Dividing by e^x gives

$$C_1'(x) + xC_2'(x) = 0,$$

so

$$C_1'(x) = -xC_2'(x) = -x(1 + 27e^{-3x} + 13e^{-x}).$$

Therefore

$$C_1'(x) = -x - 27xe^{-3x} - 13xe^{-x}.$$

Integrating,

$$C_1(x) = \int (-x - 27xe^{-3x} - 13xe^{-x}) dx = -\frac{x^2}{2} + 9xe^{-3x} + 3e^{-3x} + 13xe^{-x} + 13e^{-x}.$$

★ *Step 5:* Substitute $C_1(x)$ and $C_2(x)$ into

$$y_p(x) = C_1(x)e^x + C_2(x)xe^x.$$

After simplification we obtain

$$y_p(x) = \frac{1}{2}x^2e^x + 3e^{-2x} + 13.$$

★ *Step 6:* Finally, the general solution is obtained by adding the homogeneous and particular parts:

$$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x + 3e^{-2x} + 13$$

for every $x \in \mathbb{R}$.

Example 17.3. Consider the same equation $y'' - 2y' + y = e^x + 27e^{-2x} + 13$ as before but now we will use the Cramer's rule and the Wroskian to find the unknowns $C_1(x)$ and $C_2(x)$. As we have seen, to determine $C_1(x)$ and $C_2(x)$ we need to solve the system

$$\begin{cases} C_1'(x)e^x + C_2'(x)xe^x = 0, \\ C_1'(x)e^x + C_2'(x)(x+1)e^x = b(x), \end{cases}$$

where $b(x) = e^x + 27e^{-2x} + 13$. Its coefficient matrix is given by

$$\begin{pmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{pmatrix}.$$

The determinant of this matrix is the Wroskian of the fundamental solutions $u_1(x) = e^x$ and $u_2(x) = xe^x$, namely we have that

$$W(u_1, u_2)(x) = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}.$$

Since $W(u_1, u_2)(x) \neq 0$ for every $x \in \mathbb{R}$, the system has a unique solution. By Cramer's rule,

$$C_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ b(x) & (x+1)e^x \end{vmatrix}}{W(u_1, u_2)(x)} = \frac{-xe^x b(x)}{e^{2x}} = -\frac{x b(x)}{e^x},$$

and

$$C_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & b(x) \end{vmatrix}}{W(u_1, u_2)(x)} = \frac{e^x b(x)}{e^{2x}} = \frac{b(x)}{e^x}.$$

Now we integrate. Since

$$\frac{b(x)}{e^x} = 1 + 27e^{-3x} + 13e^{-x},$$

we get

$$C_2(x) = \int \frac{b(x)}{e^x} dx = \int (1 + 27e^{-3x} + 13e^{-x}) dx = x - 9e^{-3x} - 13e^{-x}.$$

Also,

$$C_1(x) = \int -\frac{x b(x)}{e^x} dx = \int (-x - 27xe^{-3x} - 13xe^{-x}) dx,$$

hence

$$C_1(x) = -\frac{x^2}{2} + (9x + 3)e^{-3x} + 13(x + 1)e^{-x}.$$

Therefore, one choice of the functions is

$$C_1(x) = -\frac{x^2}{2} + (9x + 3)e^{-3x} + 13(x + 1)e^{-x} \quad \text{and} \quad C_2(x) = x - 9e^{-3x} - 13e^{-x}.$$

17.2 An example without constant coefficients

Notice that the method of variation of parameters does not require the coefficients of the differential equation to be constant. In fact, the method works for general linear differential equations with variable coefficients as well. The only requirement is that we are able to find a fundamental system of solutions of the associated homogeneous equation. Once such a system $\{u_1, \dots, u_n\}$ is known, the same procedure can be applied: we replace the constants in the homogeneous solution by functions, derive a system for their derivatives, and determine them by solving that system and integrating. In practice, however, the main difficulty is that for equations with variable coefficients it is often much harder to find a fundamental system of solutions for the homogeneous equation. In the next example, we assume that we know how to find it although we do not know how.

Example 17.4. We solve

$$y'' - \frac{x}{2(x-2)}y' + \frac{1}{2(x-2)}y = \frac{x-2}{x},$$

for $x \neq 0, 2$ using the method of variation of parameters. Notice that this is not a linear equation with constant coefficients.

★ *Step 1:* Consider the homogeneous equation

$$y'' - \frac{x}{2(x-2)}y' + \frac{1}{2(x-2)}y = 0.$$

As the coefficients in this equation are not constant, we cannot use the characteristic polynomial approach. We claim instead that

$$u_1(x) = x \quad \text{and} \quad u_2(x) = e^{x/2}$$

form a fundamental system of solutions on the intervals $(-\infty, 2)$ and $(2, +\infty)$. First we check that $u_1(x) = x$ is a solution. Since $u_1'(x) = 1$ and $u_1''(x) = 0$, substituting into the equation gives

$$0 - \frac{x}{2(x-2)} + \frac{x}{2(x-2)} = 0,$$

so x is a solution. Now we check $u_2(x) = e^{x/2}$. We have

$$u_2'(x) = \frac{1}{2}e^{x/2} \quad \text{and} \quad u_2''(x) = \frac{1}{4}e^{x/2}.$$

Substituting into the equation yields

$$\frac{1}{4}e^{x/2} - \frac{x}{2(x-2)} \cdot \frac{1}{2}e^{x/2} + \frac{1}{2(x-2)} \cdot e^{x/2} = 0,$$

so $e^{x/2}$ is also a solution on $(-\infty, 2)$ and $(2, +\infty)$. To prove that they are linearly independent, we compute the Wronskian

$$W(u_1, u_2)(x) = \begin{vmatrix} x & e^{x/2} \\ 1 & \frac{1}{2}e^{x/2} \end{vmatrix} = e^{x/2} \left(\frac{x}{2} - 1 \right).$$

Since this is nonzero for $x \neq 2$, the functions x and $e^{x/2}$ are linearly independent on $(-\infty, 2)$ and on $(2, +\infty)$. Therefore they form a fundamental system and the general solution of the homogeneous equation is

$$y_h(x) = C_1x + C_2e^{x/2}.$$

★ *Step 2:* By variation of parameters we look for a particular solution of the form

$$y_p(x) = C_1(x)x + C_2(x)e^{x/2}.$$

The functions $C_1(x)$ and $C_2(x)$ satisfy the system

$$\begin{cases} C_1'(x)x + C_2'(x)e^{x/2} = 0, \\ C_1'(x) + C_2'(x)\frac{1}{2}e^{x/2} = \frac{x-2}{x}. \end{cases}$$

★ *Step 3:* Solve this system. From the first equation we obtain

$$C_1'(x) = -\frac{e^{x/2}}{x}C_2'(x).$$

Substituting this into the second equation gives

$$-\frac{e^{x/2}}{x}C_2'(x) + \frac{1}{2}e^{x/2}C_2'(x) = \frac{x-2}{x}.$$

Factoring $e^{x/2}C_2'(x)$ yields

$$e^{x/2}C_2'(x)\left(-\frac{1}{x} + \frac{1}{2}\right) = \frac{x-2}{x}.$$

Hence

$$e^{x/2}C_2'(x)\frac{x-2}{2x} = \frac{x-2}{x},$$

and therefore

$$C_2'(x) = 2e^{-x/2}.$$

Integrating,

$$C_2(x) = \int 2e^{-x/2} dx = -4e^{-x/2}.$$

★ *Step 4:* From the first equation

$$C_1'(x)x + C_2'(x)e^{x/2} = 0$$

we obtain $C_1'(x)x + 2 = 0$, so $C_1'(x) = -\frac{2}{x}$. Integrating,

$$C_1(x) = -2 \ln |x|.$$

★ *Step 5:* Substitute these expressions into $y_p(x) = C_1(x)x + C_2(x)e^{x/2}$. This gives

$$y_p(x) = -2x \ln |x| - 4.$$

★ *Step 6:* Finally, the general solution is obtained by adding the homogeneous and particular parts:

$$y(x) = y_h(x) + y_p(x) = C_1x + C_2e^{x/2} - 2x \ln |x| - 4,$$

for every $x \in \mathbb{R}$ such that $x \neq 0, 2$.