13 Lecture #9: Tuesday, March 17th, 2026

So far, we have learned how to deal with first-order differential equations, both analytically and numerically. We now turn our attention to differential equations of higher order. In general, these equations are more difficult to analyze than those of order one. However, there is an important class of higher-order equations that are significantly easier to study.

13.1 Linear differential equations

Definition 13.1. By a **linear ordinary differential equation of order** n (LODE) we mean any ODE of the form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

where $a_{n-1}, \ldots, a_0, b$ are some functions. This equation is called **homogeneous** if $b(x) = 0$. Given a linear ODE by its **associated homogeneous equation** we mean the equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

Let us see a simple examples.

Example 13.2. Consider the second–order ODE

$$y'' = xy' - 13y + e^x.$$

We can rewrite it in the standard linear form

$$y'' - xy' + 13y = e^x,$$

which shows that it is a linear ODE of order 2. On the other hand, the equation

$$y'' = \frac{y}{y'} - \sin(y) + e^x$$

is not linear, since the terms $\frac{y}{y'}$ and $\sin(y)$ involve nonlinear expressions in y and its derivatives, which are not allowed in a linear differential equation.

13.2 Structural theorems

In order to solve linear ODEs, we need the following result.

Theorem 13.3 (on existence and uniqueness for LODE)**.** Consider a linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x). \tag{L}$$

If $a_{n-1}, \ldots, a_0, b$ are continuous on an open interval I, then for all $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1} \in \mathbb{R}$ there exists a solution to the IVP (L),

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$$

on I, and it is unique there.

Notice that for an equation of order 2 we need two initial conditions. In general, for an equation of order n, we require n initial conditions.

We will prove the following result using arguments from Linear Algebra. Before doing so, let us make some observations. A differential equation can be viewed as an operator acting on functions. Instead of thinking of the equation only as a relation between derivatives, we consider it as a rule that takes a function $y(x)$ and produces another function. For example, consider the linear differential equation

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x).$$

Denote by $C^n(I)$ the vector space of all functions on the an open interval I that are n times continuously differentiable and $C(I)$ to be the vector spaces of all functions on I which are continuous. Define the operator $L : C^n(I) \to C(I)$ by

$$L[y] = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y.$$

This operator takes a function y (with sufficiently many derivatives) and produces another function. In this way, the differential equation can be written as $L[y] = b$. Thus, solving the differential equation means finding a function y that is mapped to $b(x)$ by the operator L. In particular, for the homogeneous equation $L[y] = 0$, the solutions are exactly the functions in the kernel of the operator L. This point of view connects differential equations with Linear Algebra, since L is a linear operator and the set of solutions of the homogeneous equation forms a vector space as we can see below.

Theorem 13.4 (on structure of solution set of homogeneous LODE)**.** Consider a homogeneous linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

If a_i are continuous on an open interval I, then the set of all solutions of this equation on I is a linear space of dimension n.

Proof. Let us consider the subset

$$S = \{y : I \to \mathbb{R} \mid y \text{ is in } C^n(I) \text{ and satisfies } y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0\}.$$

Observe that S is a subset of the vector space $C^n(I)$. First we show that S is a subspace of $C^n(I)$. Let $y_1, y_2 \in S$ and let $\alpha, \beta \in \mathbb{R}$. Since differentiation is linear, we have that

$$(\alpha y_1 + \beta y_2)^{(k)} = \alpha y_1^{(k)} + \beta y_2^{(k)}$$

for every $k = 1, \ldots, n$. Substituting $\alpha y_1 + \beta y_2$ into the differential equation gives

$$(\alpha y_1 + \beta y_2)^{(n)} + a_{n-1}(x)(\alpha y_1 + \beta y_2)^{(n-1)} + \cdots + a_1(x)(\alpha y_1 + \beta y_2)' + a_0(x)(\alpha y_1 + \beta y_2).$$

Using linearity we obtain

$$\alpha\big(y_1^{(n)} + a_{n-1}(x)y_1^{(n-1)} + \cdots + a_0(x)y_1\big) + \beta\big(y_2^{(n)} + a_{n-1}(x)y_2^{(n-1)} + \cdots + a_0(x)y_2\big).$$

Since $y_1, y_2 \in S$, each expression in parentheses is 0, hence $\alpha y_1 + \beta y_2$ is also a solution. Therefore S is closed under linear combinations and hence is a subspace of $C^n(I)$.

Now fix $x_0 \in I$. For each $j = 1, \dots, n$, let us the existence-uniqueness theorem for linear ODEs (see Theorem 13.3 above) to define a solution $y_j \in S$ by prescribing the initial conditions

$$y_j^{(k)}(x_0) = \begin{cases} 1, & k = j-1, \\ 0, & k \neq j-1, \end{cases} \qquad k = 0, 1, \dots, n-1.$$

That is, the vector of initial data of y_j at x_0 is the j-th standard basis vector $e_j \in \mathbb{R}^n$.

Step 1: the functions $y_1, \dots, y_n$ are linearly independent. Suppose that

$$c_1 y_1 + \cdots + c_n y_n = 0$$

as a function on I. Differentiating k times and evaluating at x_0 gives, for each $k = 0, 1, \dots, n-1$,

$$c_1 y_1^{(k)}(x_0) + \cdots + c_n y_n^{(k)}(x_0) = 0.$$

Using the defining initial conditions, the k-th equation becomes $c_{k+1} = 0$. Hence $c_1 = \cdots = c_n = 0$, proving that $\{y_1, \dots, y_n\}$ is linearly independent. Therefore, $\dim S \geq n$.

Step 2: every solution is a linear combination of $y_1, \dots, y_n$. Let $y \in S$ be arbitrary, and set $\alpha_k = y^{(k)}(x_0)$ for $k = 0, 1, \dots, n-1$. Consider the function

$$z = \alpha_0 y_1 + \alpha_1 y_2 + \cdots + \alpha_{n-1} y_n.$$

Since S is a vector space and each $y_j \in S$, we have $z \in S$. Moreover, by construction and the initial conditions of the y_j,

$$z^{(k)}(x_0) = \alpha_k = y^{(k)}(x_0)$$

for every $k = 0, 1, \dots, n-1$. Thus y and z are two solutions of the same homogeneous linear ODE having the same initial conditions at x_0. By uniqueness, $y = z$. Hence every $y \in S$ lies in $\operatorname{span}\{y_1, \dots, y_n\}$, so $\dim S \leqslant n$. Combining the two inequalities yields $\dim S = n$. ☐

We have a second structural theorem about linear ODE.

Theorem 13.5 (on structure of solution set of linear ODE)**.** Let y_p be some particular solution of a given linear ODE on an open interval I. A function y_0 is a solution of this equation on I if and only if

$$y_0 = y_p + y_h$$

for some solution y_h of the associated homogeneous equation on I. Consequently, if y_h is a general solution of the associated homogeneous equation on I, then $y_p + y_h$ is a general solution of the given equation.

Proof. Let y_p be a particular solution of the given linear ODE, that is, it satisfies

$$y_p^{(n)} + \sum_{k=0}^{n-1} a_k(x)\, y_p^{(k)} = b(x)$$

on the interval I. To simplify notation, define the linear differential operator

$$L[y] = y^{(n)} + \sum_{k=0}^{n-1} a_k(x)\, y^{(k)}.$$

We can write then the equation as $L[y] = b(x)$ and the associated homogeneous equation is $L[y] = 0$. Since this is an "if and only if" statement, we must prove both directions of the implication.

($\Rightarrow$) Suppose y_0 is a solution of the given equation on I, so $L[y_0] = b(x)$. Define $y_h := y_0 - y_p$. Using linearity of L we obtain

$$L[y_h] = L[y_0 - y_p] = L[y_0] - L[y_p] = b(x) - b(x) = 0.$$

Hence y_h is a solution of the associated homogeneous equation on I, and therefore $y_0 = y_p + y_h$.

($\Leftarrow$) Conversely, suppose that y_h is a solution of the associated homogeneous equation, i.e. $L[y_h] = 0$, and set now $y_0 := y_p + y_h$. Again by linearity,

$$L[y_0] = L[y_p + y_h] = L[y_p] + L[y_h] = b(x) + 0 = b(x),$$

so y_0 is a solution of the given equation on I. This proves that y_0 solves $L[y] = b(x)$ if and only if $y_0 = y_p + y_h$ for some homogeneous solution y_h. □

Theorem 13.5 says that, in practice, the procedure goes as follows.

1. Find one particular solution y_p of the equation $L[y] = b(x)$.
2. Find the general solution y_h of the associated homogeneous equation $L[y] = 0$.
3. Add them to obtain the general solution of the original equation: $y = y_p + y_h$.

For example, consider $y'' - y = e^x$. Suppose we know a way of finding the particular solution $y_p = \frac{1}{2}xe^x$ and also that we know a way of finding the general solution of the homogeneous equation $y'' - y = 0$ which is given by $y_h = C_1e^x + C_2e^{-x}$. Then the general solution of the original equation must be

$$y(x) = \tfrac{1}{2}xe^x + C_1e^x + C_2e^{-x}$$

where $C_1, C_2 \in \mathbb{R}$, by using the ideas of Theorem 13.5.

13.3 Homogeneous equations

The first step we need to deal with is how to solve the homogeneous equation associated a linear ODE.

Definition 13.6. Consider a homogeneous linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0.$$

Assume that a_i are continuous on an open interval I. By a **fundamental system of solutions** of this equation on I we mean any basis of the space of all solutions of this equation on I.

A fundamental system of solutions of a homogeneous linear ODE of order n on an interval I is a set of n solutions that are linearly independent on I. Since the solution set is an n–dimensional vector space, such a set forms a basis, so every solution can be written uniquely as a linear combination of the functions in the fundamental system. For instance, consider the equation

$$y'' - 3y' + 2y = 0.$$

Let us check that $\{e^x, e^{2x}\}$ is a fundamental system of solutions of this equation. First of all, let us check that $y_1(x) = e^x$ and $y_2(x) = e^{2x}$ are solutions. For $y_1 = e^x$, we have $y_1' = e^x$ and $y_1'' = e^x$, hence

$$y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0.$$

So y_1 is a solution. Likewise, if $y_2 = e^{2x}$, then we have $y_2' = 2e^{2x}$ and $y_2'' = 4e^{2x}$, hence

$$y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3(2e^{2x}) + 2e^{2x} = 4e^{2x} - 6e^{2x} + 2e^{2x} = 0.$$

To check linear independence, assume there exist constants $c_1, c_2 \in \mathbb{R}$ such that $c_1e^x + c_2e^{2x} = 0$ for all $x \in I$. Factoring out e^x (which is never zero), we get $e^x\big(c_1 + c_2e^x\big) = 0$ again for all $x \in I$. So,

$$c_1 + c_2e^x = 0 \qquad \text{for all } x \in I.$$

Choose two distinct points $x_1, x_2 \in I$ with $x_1 \neq x_2$. Then

$$c_1 + c_2e^{x_1} = 0 \quad \text{and} \quad c_1 + c_2e^{x_2} = 0.$$

Subtracting gives

$$c_2\big(e^{x_1} - e^{x_2}\big) = 0.$$

Since $e^{x_1} \neq e^{x_2}$, we obtain $c_2 = 0$, and then $c_1 = 0$. Hence e^x and e^{2x} are linearly independent on I. Therefore, $\{e^x, e^{2x}\}$ is a fundamental system of solutions of $y'' - 3y' + 2y = 0$ on I.

In practice, checking that a set of solutions is linearly independent by working directly with the definition can be cumbersome. Fortunately, there is a convenient tool called the Wronskian that allows us to verify linear independence more easily.

Definition 13.7. Let $y_1, y_2, \ldots, y_n$ be $(n-1)$-times differentiable functions. We define their **Wronskian** as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Let $y_1(x) = e^x$ and $y_2(x) = e^{2x}$. Their derivatives are

$$y_1'(x) = e^x, \qquad y_2'(x) = 2e^{2x}.$$

The Wronskian of y_1 and y_2 is defined as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Substituting the functions and their derivatives gives

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}.$$

Computing the determinant,

$$W(x) = e^x \cdot 2e^{2x} - e^{2x} \cdot e^x = 2e^{3x} - e^{3x} = e^{3x}.$$

It turns that the only thing we need to do to show that $\{y_1(x), y_2(x)\}$ are linearly independent is to show that the Wronskian is different from zero.

Theorem 13.8. Consider a homogeneous linear ODE

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0,$$

where a_i are continuous on an open interval I. Let $y_1, y_2, \ldots, y_n$ be solutions of this equation on I, and let W be their Wronskian. These functions form a linearly independent set (and thus a fundamental system) if and only if

$$W(x) \neq 0 \quad \text{on } I,$$

which is equivalent to the existence of some $x_0 \in I$ such that

$$W(x_0) \neq 0.$$

The previous example shows that $W(x) = e^{3x}$, which is never zero. This implies that the functions $\{e^x, e^{2x}\}$ are linearly independent and therefore form a fundamental system of solutions. Consequently, the general solution of the equation $y'' - 3y' + 2y = 0$ is

$$y(x) = c_1e^x + c_2e^{2x}$$

for $c_1, c_2 \in \mathbb{R}$.

We will be working with very specific kind of linear equation. Indeed, we will be working with the ones who have constant coefficients.

Definition 13.9. By a **linear ODE with constant coefficients** we mean any linear ODE for which $a_0(x) = a_0$, $a_1(x) = a_1$, …, $a_{n-1}(x) = a_{n-1}$ are constant functions.

We need to learn some new names before trying to solve these equations.

Definition 13.10. Consider a homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

We define its **characteristic polynomial** as

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

We define its **characteristic equation** as $p(\lambda) = 0$. The solutions of this equation are called **characteristic numbers** or **eigenvalues** of the given ODE.

Consider once again the equation $y'' - 3y' + 2y = 0$. The associated characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, and therefore the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$. The eigenvalues of this equation are $\lambda_1 = 1$ and $\lambda_2 = 2$ as one can see by solving the characteristic equation. Recall that we previously claimed that the general solution of this equation is determined by the fundamental system $\{e^{1\cdot x}, e^{2\cdot x}\}$. One may wonder whether the numbers 1 and 2 appearing in the exponents are merely a coincidence with the eigenvalues of the characteristic equation. The answer is no, as the following result shows.

Fact 13.11. Consider a homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

Let λ_0 be its characteristic number. Then $y(x) = e^{\lambda_0 x}$ is a solution of this equation. If $\lambda_1, \ldots, \lambda_N$ are distinct characteristic numbers of this equation, then $\{e^{\lambda_1 x}, \ldots, e^{\lambda_N x}\}$ is a linearly independent set of solutions.

Proof. Let us prove just the first part of this result. Consider the homogeneous linear ODE with constant coefficients $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$ and define the linear differential operator $L[y] = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y$. Let λ_0 be a characteristic number, i.e.

$$p(\lambda_0) = \lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0 = 0.$$

Set $y_0(x) = e^{\lambda_0 x}$. Since $(e^{\lambda_0 x})^{(k)} = \lambda_0^k e^{\lambda_0 x}$ for every k, we obtain

$$L[y_0] = L\big[e^{\lambda_0 x}\big] = \Big(\lambda_0^n + a_{n-1}\lambda_0^{n-1} + \cdots + a_1\lambda_0 + a_0\Big)e^{\lambda_0 x} = p(\lambda_0)e^{\lambda_0 x} = 0.$$

Hence $y_0(x) = e^{\lambda_0 x}$ is a solution of the equation. □

In practice, to solve a homogeneous linear ODE with constant coefficients, we proceed as follows. First, we write the equation in the standard form $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$. Then we form the characteristic polynomial $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ and solve the characteristic equation $p(\lambda) = 0$. Each root λ_i of this equation produces a solution $y_i(x) = e^{\lambda_i x}$. If the roots $\lambda_1, \ldots, \lambda_n$ are distinct, then the functions $\{e^{\lambda_1 x}, \ldots, e^{\lambda_n x}\}$ are linearly independent and form a fundamental system of solutions. Finally, the general solution of the differential equation is obtained by taking a linear combination of these functions, $y(x) = c_1e^{\lambda_1 x} + \cdots + c_ne^{\lambda_n x}$, where $c_1, \ldots, c_n$ are arbitrary constants.

What happens if the characteristic numbers $\lambda_1, \ldots, \lambda_N$ are not distinct? Let us see the following example.

Example 13.12. We consider the Cauchy problem

$$\begin{cases} y^{(4)} - 3y'' + 2y' = 0, \\ y(0) = 3, \\ y'(0) = -6, \\ y''(0) = 13, \\ y'''(0) = -22. \end{cases}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^4 - 3\lambda^2 + 2\lambda = \lambda(\lambda^3 - 3\lambda + 2) = \lambda(\lambda - 1)(\lambda^2 + \lambda - 2) = \lambda(\lambda - 1)^2(\lambda + 2).$$

Hence the characteristic numbers are $\lambda = 0$, $\lambda = 1$ (double root), and $\lambda = -2$. One might be tempted to conclude that the fundamental system is given by $\{1, e^x, e^{-2x}\}$. However, this cannot be correct, since a fourth–order linear ODE requires four linearly independent solutions. Thus, we must find an additional solution to complete the fundamental system and this is solved in the next result.

Fact 13.13. Consider a homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

Let λ_0 be its characteristic number with multiplicity m. Then, $e^{\lambda_0 x}$, $xe^{\lambda_0 x}$, ..., $x^{m-1}e^{\lambda_0 x}$ are solutions of this equation and they form a linearly independent set.

Let us go back to Example 13.12. From there and using the above fact, we have that the fundamental system is given by

$$\{1, e^x, x \cdot e^x, e^{-2x}\}$$

and therefore the general solution is given by

$$y(x) = C_1 + C_2e^x + C_3xe^x + C_4e^{-2x} = C_1 + (C_2 + C_3x)e^x + C_4e^{-2x}$$

for every $x \in \mathbb{R}$, where $C_1, C_2, C_3, C_4 \in \mathbb{R}$. Notice that we are not done yet. We have to deal with the initial conditions still. For this, we compute derivatives

$$y'(x) = (C_2 + C_3x + C_3)e^x - 2C_4e^{-2x},$$

$$y''(x) = (C_2 + C_3x + 2C_3)e^x + 4C_4e^{-2x},$$
$$y'''(x) = (C_2 + C_3x + 3C_3)e^x - 8C_4e^{-2x}.$$

Evaluating at $x = 0$ gives the system

$$\begin{cases} C_1 + C_2 + C_4 = 3, \\ C_2 + C_3 - 2C_4 = -6, \\ C_2 + 2C_3 + 4C_4 = 13, \\ C_2 + 3C_3 - 8C_4 = -22. \end{cases}$$

Solving yields $C_1 = 1$, $C_2 = -1$, $C_3 = 1$, $C_4 = 3$. Therefore, the solution of the Cauchy problem is

$$y(x) = 1 + (x - 1)e^x + 3e^{-2x}$$

for every $x \in \mathbb{R}$.