NUM: Solved problems—numerical analysis

1. a) Find an approximating formula for the function $f(x) = \arctan(x)$ on a neighborhood of $a = 1$ with error $O(h^4)$.
b) Estimate $\arctan(1.2)$ using quadratic approximation.

2. Estimate $\int\limits_0^4 \frac{1}{2}x\,dx$ using the rectangle and the trapezoid methods with partition size $n = 2$.

3. Consider the equation $\frac{1}{x} = x^2 + 1$.
a) Using the Newton method, deduce an iterating scheme that could converge to the solution of this equation. Show the first two steps of this iteration for the case $x_0 = 1$.
b) Transform the equation into a fixed point problem and create an iterating formula. Show the first two steps of this iteration for the case $x_0 = 2$.
Inquire whether the behaviour around x_0 looks hopeful.
Then set up a general relaxed version of this iteration and find the optimal relaxation parameter λ for a neighborhood of $x_0 = 2$.

4. Consider the initial value problem $y' = 1 + x^2 y$, $y(-2) = 1$. Use the Euler method to set up iterating formulas for finding an approximate solution on the interval $[-2, 6]$ with step size $h = 2$. Calculate the first three points.

5. Consider the system
$$\left[\begin{array}{rl} x - y &= 4,\\ x + y + 3z &= 9,\\ 2x + z &= 7. \end{array}\right.$$
a) Solve it (and comment on your steps) using the Gaussian elimination and the back substitution.
b) Using the Gauss-Seidel method, create an iterating scheme for solving this system. Show the first two iterations for the case $\vec{x}_0 = (1, 1, 1)$.

Solutions

1. $f(x) = \arctan(x)$, $f'(x) = \frac{1}{x^2+1}$, $f''(x) = \frac{-2x}{(x^2+1)^2}$, $f'''(x) = \frac{6x^2-2}{(x^2+1)^3}$.
Hence $f(1) = \frac{\pi}{4}$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{2}$, $f'''(1) = \frac{1}{2}$.
Version 1: a) $f(a+h) = f(a) + f'(a)h + \frac{1}{2!}f''(a)h^2 + \dots + \frac{1}{n!}f^{(n)}(a)h^n + O(h^{n+1})$.
Here $a = 1$, so $\arctan(1+h) \approx \frac{\pi}{4} + \frac{1}{2}h - \frac{1}{4}h^2 + \frac{1}{12}h^3 + O(h^4)$.

Version 2: $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \dots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + O((x-a)^{n+1})$.
Here $a = 1$, so $\arctan(x) \approx \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + O((x-1)^4)$.

Version 3: Auxiliary function $g(h) = f(a+h) = \arctan(1+h)$.
Then $g'(h) = \frac{1}{(h+1)^2+1}$, $g''(h) = \frac{-2(h+1)}{((h+1)^2+1)^2}$, $g'''(h) = \frac{6(h+1)^2-2}{((h+1)^2+1)^3}$.
Hence $g(0) = \frac{\pi}{4}$, $g'(0) = \frac{1}{2}$, $g''(0) = -\frac{1}{2}$, $g'''(0) = \frac{1}{2}$.
$f(a+h) = g(h) \approx g(0) + g'(0)h + \frac{1}{2!}g''(0)h^2 + \dots + \frac{1}{n!}g^{(n)}(0)h^n + O(h^{n+1})$, so
$\arctan(1+h) \approx \frac{\pi}{4} + \frac{1}{2}h - \frac{1}{4}h^2 + \frac{1}{12}h^3 + O(h^4)$.

b) By the above calculations, $\arctan(1+h) \approx \frac{\pi}{4} + \frac{1}{2}h - \frac{1}{4}h^2$, hence
$\arctan(1.2) \approx \frac{\pi}{4} + \frac{1}{2}\cdot 0.2 - \frac{1}{4}\cdot 0.2^2 = \frac{\pi}{4} + 0.09 \approx 0.875...$

2. From $n = 2$ we get $h = \frac{4-0}{2} = 2$. Points $x_0 = 0$, $x_1 = 2$, $x_2 = 4$.
There are two rectangular methods, with the left and right rectangles.
Left: $R_l(2) = 2\cdot(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2) = 2$.
Right: $R_r(2) = 2\cdot(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 4) = 6$.
Trapezoid: $T(2) = \frac{1}{2}\cdot 2\cdot(\frac{1}{2}\cdot 0 + 2\cdot\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 4) = 4$.

3. a) Rewrite as a root problem: $\frac{1}{x} - x^2 - 1 = 0$. So $f(x) = \frac{1}{x} - x^2 - 1$. Scheme:
$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{\frac{1}{x_k} - x_k^2 - 1}{-\frac{1}{x_k^2} - 2x_k} = \frac{x_k^4 - x_k^2 + 2x_k}{1 + 2x_k^3}$. $\qquad x_0 = 1$, $x_1 = \frac{2}{3}$, $x_2 = \frac{\frac{16}{81} - \frac{4}{9} + \frac{4}{3}}{1 + \frac{16}{81}} = \frac{88}{129}$.

b) Standard: First $\frac{1}{x}-x^2-1=0$, then $\frac{1}{x}-x^2-1+x=x$, so $\varphi(x)=\frac{1}{x}-x^2-1+x$.
Iterative formula: $x_{k+1}=\frac{1}{x_k}-x_k^2-1+x_k$. $x_0=2$, $x_1=-\frac{5}{2}$, $x_2=-\frac{203}{20}$, ...
Convergence is usually judged through contraction, in particular we want $|\varphi'(x)|<1$.
$\varphi'(x)=-\frac{1}{x^2}-2x+1$, $|\varphi'(2)|=\frac{13}{4}>1$. This does not look hopeful. However, the first iteration already leads us someplace else (large negative numbers), where the situation may perhaps be different. Unfortunately it is not, φ' is large there as well.
Relaxed version: $x_{k+1}=\lambda\big(\frac{1}{x_k}-x_k^2-1+x_k\big)+(1-\lambda)x_k$.
Optimalisation: $\varphi_\lambda=\lambda\big(\frac{1}{x}-x^2-1+x\big)+(1-\lambda)x$ We want $\varphi'(x_0)=0$.
The equation $\lambda\big(\frac{-1}{x^2}-2x+1\big)+(1-\lambda)=0$ yields $\lambda=\frac{1}{1+\frac{1}{x^2}+2x-1}$, for $x=2$ we get $\lambda_{\text{opt}}=\frac{4}{17}$.

Alternative: $x=\frac{1}{x^2+1}$, so $\varphi(x)=\frac{1}{x^2+1}$.
Iterative formula: $x_{k+1}=\frac{1}{x_k^2+1}$. $x_0=2$, $x_1=\frac{1}{5}$, $x_2=\frac{25}{26}$, ...
Convergence: $\varphi'(x)=\frac{-2x}{(x^2+1)^2}$, $|\varphi'(2)|=\frac{4}{25}<1$. This looks hopeful. Iterations look promising, as if they were jumping from side to side, but ever closer to some middle value that perhaps is the number we want.
More detailed analysis: φ maps $[0,2]$ to $[0,1)\subseteq[0,2]$, on this set we have $|\varphi'|<1$, so by the Banach contraction theorem we should get convergence.
Relaxed version: $x_{k+1}=\lambda\frac{1}{x_k^2+1}+(1-\lambda)x_k$.
Optimalisation: $\varphi_\lambda=\lambda\frac{1}{x^2+1}+(1-\lambda)x$ We want $\varphi'(x_0)=0$.
The equation $\lambda\frac{-2x}{(x^2+1)^2}+(1-\lambda)=0$ yields $\lambda=\frac{1}{1+\frac{2x}{(x^2+1)^2}}$, for $x=2$ we get $\lambda_{\text{opt}}=\frac{25}{29}$.

Alternative: $x=\sqrt{\frac{1}{x}-1}$, so $\varphi(x)=\sqrt{\frac{1}{x}-1}$.
Iterative formula: $x_{k+1}=\sqrt{\frac{1}{x_k}-1}$. $x_0=2$, $x_1=\sqrt{-\frac{1}{2}}$
And we have a problem. This does not look like a good idea. However, if we start with $x_0<1$ then the square root does not cause trouble. Then we also have $\varphi'(x)=\frac{-\frac{1}{x^2}}{2\sqrt{\frac{1}{x}-1}}$, for instance $\big|\varphi'\big(\frac{1}{2}\big)\big|=2>1$, this is not very hopeful.
Relaxed version: $x_{k+1}=\lambda\sqrt{\frac{1}{x_k}-1}+(1-\lambda)x_k$.
Optimalisation: $\varphi_\lambda=\lambda\sqrt{\frac{1}{x}-1}+(1-\lambda)x$ We want $\varphi'(x_0)=0$.
The equation $\lambda\frac{-1}{2x^2\sqrt{\frac{1}{x}-1}}+(1-\lambda)=0$ yields $\lambda=\frac{1}{1+\frac{1}{2x^2\sqrt{\frac{1}{x}-1}}}$, for $x=2$ we cannot substitute.

Alternative: $x^2=\frac{1}{x}-1$ neboli $x=\frac{1}{x^2}-\frac{1}{x}$, so $\varphi(x)=\frac{1}{x^2}-\frac{1}{x}=\frac{1-x}{x^2}$.
Iterative formula: $x_{k+1}=\frac{1-x_k}{x_k^2}$. $x_0=2$, $x_1=-\frac{1}{4}$, $x_2=20$, ...
This does not look good. Analysis: $\varphi'(x)=\frac{x-2}{x^3}$, then $|\varphi'(2)|=0$. This looks hopeful, but the iterations quickly run elsewhere, for isntance some are close to zero where the derivative is large. When we look at the formula, we see that is turns large numbers into small and vice versa. This looks like an unpleasant oscillation.
Relaxed version: $x_{k+1}=\lambda\frac{1-x_k}{x_k^2}+(1-\lambda)x_k$.
Optimalisation: $\varphi_\lambda=\lambda\frac{1-x}{x^2}+(1-\lambda)x$ We want $\varphi'(x_0)=0$.
The equation $\lambda\frac{x-2}{x^3}+(1-\lambda)=0$ yields $\lambda=\frac{1}{1-\frac{x-2}{x^3}}$, for $x=2$ we get $\lambda_{\text{opt}}=1$.

4. The main iterating formula is $y_{k+1}=y_k+h\cdot f(x_k,y_k)$. Step size $h=2$ was given, so partition size is $n=4$.
Scheme:
(0) $x_0=-2$, $y_0=1$.
(1) $x_{k+1}=x_k+2$, $y_{k+1}=y_k+2\cdot(1+x_k^2y_k)$ for $i=0,..,3$, that is, $i=0,1,2,3$.
The iterating formula can be simplified as $y_{k+1}=2+(1+2x_k^2)y_k$, wher y_k is used only once now.
Points: $(-2,1)$, $y_1=2+(1+2\cdot4)\cdot1$, so $(0,11)$; $y_2=2+(1+2\cdot0)\cdot11$, so $(2,13)$.

5. a) Step 1 (GEM): Extended matrix of the system $\begin{pmatrix} 1 & -1 & 0 & 4 \\ 1 & 1 & 3 & 9 \\ 2 & 0 & 1 & 7 \end{pmatrix}$ is modified using the Gaussian elimination to an upper-triangular form as follows:

$$\begin{pmatrix} 1 & -1 & 0 & 4 \\ 1 & 1 & 3 & 9 \\ 2 & 0 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 2 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & -2 & -6 \end{pmatrix}.$$

Step 2 (BS): The resulting system of equations $\left[\begin{aligned} x - y \quad &= 4 \\ 2y + 3z &= 5 \\ -2z &= -6 \end{aligned}\right.$ is solved from the last to the first: $z = 3$, $y = \frac{1}{2}(5 - 3z) = -2$, $x = y + 4 = 2$.

b) We rewrite the system as $\left[\begin{aligned} x &= 4 + y, \\ y &= 9 - x - 3z, \\ z &= 7 - 2x. \end{aligned}\right.$ These are the iterating equations. If we use the latest values, we get the Gauss-Seidel iteration. Formally:

$$\left[\begin{aligned} x_{k+1} &= 4 + y_k, \\ y_{k+1} &= 9 - x_{k+1} - 3z_k, \\ z_{k+1} &= 7 - 2x_{k+1} \end{aligned}\right. \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 4 + 1 = 5 \\ 9 - 5 - 3 \cdot 1 = 1 \\ 7 - 2 \cdot 5 = -3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 4 + 1 = 5 \\ 9 - 5 - 3 \cdot (-3) = 13 \\ 7 - 2 \cdot 5 = -3 \end{pmatrix}$$

Remark: The matrix does not look like a diagonally dominant one, so the convergence does not look hopeful. It is better to change the order of equations:

$$\left[\begin{aligned} 2x \quad + z &= 7 \\ x - y \quad &= 4 \\ x + y + 3z &= 9 \end{aligned}\right. \Longrightarrow \left[\begin{aligned} x_{k+1} &= \tfrac{7}{2} - \tfrac{1}{2} z_k \\ y_{k+1} &= x_{k+1} - 4 \\ z_{k+1} &= 3 - \tfrac{1}{3} x_{k+1} - \tfrac{1}{3} y_{k+1}. \end{aligned}\right. \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 3 \\ -1 \\ \frac{7}{3} \end{pmatrix} \Longrightarrow \begin{pmatrix} \frac{7}{3} \\ -\frac{5}{3} \\ \frac{25}{9} \end{pmatrix}.$$