

## 12 Practice #4: Wednesday, March 11th, 2026

### 12.1 The Euler method and predicting error using order

#### 12.2 Problem 1

Consider the IVP  $y' = f(x, y)$  with  $y(x_0) = y_0$  on an interval  $[x_0, x_0 + T]$ , and choose a partition size  $n \in \mathbb{N}$  with step size  $h = \frac{T}{n}$ . The Euler method constructs a sequence  $(x_i, y_i)$  by considering the initial value  $(x_0, y_0)$  and for by taking for each  $i = 0, 1, \dots, n - 1$ ,

$$x_{i+1} = x_i + h \quad \text{and} \quad y_{i+1} = y_i + h f(x_i, y_i).$$

The Euler method is of order 1, meaning that the error satisfies an estimate of the form

$$E_h \sim c \cdot h$$

for some constant  $c > 0$ . Now consider the equation  $y' = -x(y - 1)$  with initial value  $y(0) = 2$ , on the interval  $[0, 5]$  with partition size  $n = 10$ . In this case, the step size is given by

$$h = \frac{5 - 0}{10} = 0.5.$$

Also, for this problem, we have that  $f(x, y) = -x(y - 1)$ . Starting from  $(x_0, y_0) = (0, 2)$ , we compute the first three iterates: For  $i = 0$ , we have that

$$x_1 = x_0 + h = 0.5 \quad \text{and} \quad y_1 = y_0 + h f(x_0, y_0) = 2 + 0.5(-0(2 - 1)) = 2.$$

For  $i = 1$ , we have that

$$x_2 = x_1 + h = 1.0 \quad \text{and} \quad y_2 = y_1 + h f(x_1, y_1) = 2 + 0.5(-0.5(2 - 1)) = 2 - 0.25 = 1.75.$$

Thus, the first three points of the iteration are

$$(x_0, y_0) = (0, 2), \quad (x_1, y_1) = (0.5, 2) \quad \text{and} \quad (x_2, y_2) = (1.0, 1.75).$$

#### 12.3 Problem 2

Consider the IVP  $y' = x + y^2$  with  $y(1) = 0$  on the interval  $[1, 7]$  with partition size  $n = 3$ . In this case, the step size is

$$h = \frac{7 - 1}{3} = 2.$$

Euler's method is given by

$$x_{i+1} = x_i + h \quad \text{and} \quad y_{i+1} = y_i + h f(x_i, y_i)$$

where  $f(x, y) = x + y^2$ . Starting from  $(x_0, y_0) = (1, 0)$ , we get the following next points. For  $i = 0$ , we have that

$$x_1 = x_0 + h = 1 + 2 = 3 \quad \text{and} \quad y_1 = y_0 + h f(x_0, y_0) = 0 + 2(1 + 0^2) = 2.$$

For  $i = 1$ , we have that

$$x_2 = x_1 + h = 3 + 2 = 5 \quad \text{and} \quad y_2 = y_1 + h f(x_1, y_1) = 2 + 2(3 + 2^2) = 2 + 2(7) = 16.$$

Thus, the first three elements of the iteration are

$$(x_0, y_0) = (1, 0), \quad (x_1, y_1) = (3, 2) \quad \text{and} \quad (x_2, y_2) = (5, 16).$$

## 12.4 Problem 3

Consider the IVP  $y' = f(x, y)$  with  $y(x_0) = y_0$  on an interval  $[x_0, x_0 + T]$ , and choose a partition size  $n \in \mathbb{N}$  with step size  $h = \frac{T}{n}$ . The Heun method constructs a sequence  $(x_i, y_i)$  by taking the initial value  $(x_0, y_0)$  and for each  $i = 0, 1, \dots, n - 1$ ,

- ★  $x_{i+1} = x_i + h$ ,
- ★  $k_1 = f(x_i, y_i)$ ,
- ★  $y_{i+1}^* = y_i + h k_1$ ,
- ★  $k_2 = f(x_{i+1}, y_{i+1}^*)$  and finally
- ★  $y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2) \cdot h$ .

The Heun method is of order 2, meaning that the error satisfies an estimate of the form

$$E_h \sim c \cdot h^2$$

for some constant  $c > 0$ . Now consider  $y' = -x(y - 1)$  with  $y(0) = 2$  on the interval  $[0, 5]$  with partition size  $n = 10$ . Then, the step size is given by

$$h = \frac{5 - 0}{10} = 0.5$$

and, in this case,  $f(x, y) = -x(y - 1)$ . Starting from  $(x_0, y_0) = (0, 2)$ , we compute the first step. For  $i = 0$ , we get that

- ★  $x_1 = 0.5$ ,
- ★  $k_1 = f(0, 2) = -0(2 - 1) = 0$ ,
- ★  $y_1^* = 2 + 0.5 \cdot 0 = 2$ ,
- ★  $k_2 = f(0.5, 2) = -0.5(2 - 1) = -0.5$  and
- ★  $y_1 = 2 + \frac{0.5}{2}(0 - 0.5) = 2 + 0.25(-0.5) = 1.875$ .

Thus, the first two points of the iteration are

$$(x_0, y_0) = (0, 2) \quad \text{and} \quad (x_1, y_1) = (0.5, 1.875).$$

## 12.5 Problem 4

Consider the IVP  $y' = f(x, y)$  with  $y(x_0) = y_0$  on an interval  $[x_0, x_0 + T]$ , and choose a partition size  $n \in \mathbb{N}$  with step size  $h = \frac{T}{n}$ . The midpoint method (RK2) constructs a sequence  $(x_i, y_i)$  by taking the initial point  $(x_0, y_0)$  and for each  $i = 0, 1, \dots, n - 1$ ,

- ★  $x_{i+1} = x_i + h$ ,

- ★  $k_1 = f(x_i, y_i)$ ,
- ★  $y_{i+\frac{1}{2}}^* = y_i + \frac{h}{2} \cdot k_1$ ,
- ★  $k_2 = f\left(x_i + \frac{h}{2}, y_{i+\frac{1}{2}}^*\right)$  and finally
- ★  $y_{i+1} = y_i + h \cdot k_2$ .

The midpoint method is of order 2, meaning that the error satisfies an estimate of the form

$$E_h \sim c \cdot h^2$$

for some constant  $c > 0$ . Now consider  $y' = -x(y - 1)$  with  $y(0) = 2$  on the interval  $[0, 5]$  with partition size  $n = 10$ . Then, once again,

$$h = \frac{5 - 0}{10} = 0.5.$$

Starting from  $(x_0, y_0) = (0, 2)$ , we compute the first step. For  $i = 0$ , we have that

- ★  $x_1 = 0.5$ ,
- ★  $k_1 = f(0, 2) = -0(2 - 1) = 0$ ,
- ★  $y_{1/2}^* = 2 + \frac{0.5}{2} \cdot 0 = 2$ ,
- ★  $k_2 = f(0.25, 2) = -0.25(2 - 1) = -0.25$  and
- ★  $y_1 = 2 + 0.5(-0.25) = 1.875$

Thus, the first two points of the iteration are

$$(x_0, y_0) = (0, 2) \quad \text{and} \quad (x_1, y_1) = (0.5, 1.875).$$

Notice that it is merely a coincidence that the first two points obtained with the midpoint method coincide with those from the Heun method in this example; in general, these two methods produce different approximations.

## 12.6 Problem 5

Suppose that we have solved an IVP numerically using the midpoint method with step size  $h = 0.5$ , and we obtained an estimated error

$$E_{0.5} \approx 0.01888.$$

Recall that the midpoint method is a method of order  $p = 2$ , so the error behaves approximately like

$$E_h \approx ch^2$$

for some constant  $c > 0$ .

(a) If we reduce the step size to  $\frac{1}{2} \cdot 0.5 = 0.25$ , then

$$E_{0.25} \approx E_{0.5} \left( \frac{0.25}{0.5} \right)^2 = 0.01888 \left( \frac{1}{2} \right)^2 = 0.01888 \cdot \frac{1}{4} = 0.00472.$$

(b) We now want to find a step size  $\tilde{h}$  such that

$$E_{\tilde{h}} = 0.00001.$$

Using the error model,

$$0.00001 \approx 0.01888 \left( \frac{\tilde{h}}{0.5} \right)^2.$$

Thus,

$$\left( \frac{\tilde{h}}{0.5} \right)^2 = \frac{0.00001}{0.01888} \approx 0.00052966,$$

and therefore

$$\frac{\tilde{h}}{0.5} \approx \sqrt{0.00052966} \approx 0.0230.$$

Hence

$$\tilde{h} \approx 0.5 \cdot 0.0230 \approx 0.0115.$$

Therefore, a recommended step size is

$$\tilde{h} \approx 0.0115.$$

Another approach to solve item (b). Recall that for a method of order  $p = 2$  we have

$$E_{\frac{h}{a}} \approx \frac{1}{a^2} E_h.$$

In this problem we know that  $E_{0.5} \approx 0.01888$  and we want

$$E_{\frac{0.5}{a}} = 0.00001.$$

Thus,

$$0.00001 \approx \frac{1}{a^2} 0.01888.$$

Solving for  $a$ ,

$$a^2 \approx \frac{0.01888}{0.00001} = 1888, \quad a \approx \sqrt{1888} \approx 43.45.$$

Therefore, the new step size is

$$\tilde{h} = \frac{0.5}{a} \approx \frac{0.5}{43.45} \approx 0.0115.$$

Hence, a recommended step size is

$$\tilde{h} \approx 0.0115.$$

## 12.7 Problem 6

Assume the numerical method has order  $p$ , so that for sufficiently small  $h$  its error satisfies

$$E_h \approx c \cdot h^p$$

for some constant  $c > 0$ . If we reduce the step size by a factor  $a > 0$ , then

$$E_{h/a} \approx C \left(\frac{h}{a}\right)^p = C h^p \frac{1}{a^p}.$$

Dividing the two error expressions gives

$$\frac{E_h}{E_{h/a}} \approx \frac{C h^p}{C h^p / a^p} = a^p.$$

Taking logarithms (base  $a$ ) on both sides,

$$\log_a \left( \frac{E_h}{E_{h/a}} \right) \approx \log_a (a^p) = p,$$

and therefore

$$p \approx \log_a \left( \frac{E_h}{E_{h/a}} \right).$$

In words, the calculation shows that the order of the method can be estimated by observing how much the error decreases when the step size is reduced.