15 Practice #5: Wednesday, March 18th, 2026

15.1 Linear ODEs - homogeneous case

15.2 Problem 1

Consider the homogeneous differential equation

$$y''' + 4y'' + 4y' = 0.$$

(a) We will find the general solution and then

(b) describe the asymptotic behavior of the solutions as $x \to +\infty$.

Since this is a linear ODE with constant coefficients, we look for solutions of the form $y = e^{\lambda x}$. The characteristic polynomial is

$$p(\lambda) = \lambda^3 + 4\lambda^2 + 4\lambda = \lambda(\lambda^2 + 4\lambda + 4) = \lambda(\lambda + 2)^2.$$

Hence the characteristic numbers are $\lambda = 0$ and $\lambda = -2$ (with multiplicity 2). Therefore, a fundamental set of solutions is $\{1, e^{-2x}, x \cdot e^{-2x}\}$ and then the general solution is given by

$$y(x) = C_1 + C_2 e^{-2x} + C_3 x e^{-2x}$$

for every $x \in \mathbb{R}$ where $C_1, C_2, C_3 \in \mathbb{R}$ are constants. This answers item (a). Let us now answer (b). As $x \to +\infty$, we have $e^{-2x} \to 0$ and also $xe^{-2x} \to 0$. Therefore, every solution satisfies $y(x) \to C_1 \quad \text{as } x \to +\infty$. In particular, all solutions are bounded, and they converge to a constant. Moreover, if $C_1 = 0$, then $y(x)$ decays to 0 exponentially fast since both remaining terms go to 0.

15.3 Problem 2

Consider the homogeneous differential equation

$$y^{(5)} - 4y''' = 0.$$

(a) We will once again find the general solution and then

(b) describe the asymptotic behavior of the solutions as $x \to +\infty$.

We look for solutions of the form $y = e^{\lambda x}$. The characteristic polynomial is

$$p(\lambda) = \lambda^5 - 4\lambda^3 = \lambda^3(\lambda^2 - 4) = \lambda^3(\lambda - 2)(\lambda + 2).$$

Hence the characteristic numbers are $\lambda = 0$ with multiplicity 3, $\lambda = 2$ and $\lambda = -2$. Therefore, a fundamental set of solutions is $\{e^0, e^0 \cdot x, e^0 \cdot x^2, e^{2x}, e^{-2x}\} = \{1, x, x^2, e^{2x}, e^{-2x}\}$ and the general solution is given by

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{2x} + C_5 e^{-2x}$$

for every $x \in \mathbb{R}$ where $C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}$. This gives the answer to item (a). Now we will answer item (b). As $x \to +\infty$, the term $e^{-2x} \to 0$, while $e^{2x} \to +\infty$. The polynomial part $C_1 + C_2 x + C_3 x^2$ grows at most quadratically, which is negligible compared to e^{2x} when $C_4 \neq 0$. We have the following cases.

- ⋆ If $C_4 \neq 0$, then $y(x) \sim C_4 e^{2x}$, so $y(x) \to +\infty$ if $C_4 > 0$ and $y(x) \to -\infty$ if $C_4 < 0$.
- ⋆ If $C_4 = 0$ and $C_3 \neq 0$, then $y(x) \sim C_3 x^2$, so $|y(x)| \to +\infty$ with polynomial growth.
- ⋆ If $C_4 = 0$ and $C_3 = 0$ but $C_2 \neq 0$, then $y(x) \sim C_2 x$, so $|y(x)| \to +\infty$ linearly.
- ⋆ If $C_4 = C_3 = C_2 = 0$, then $y(x) \to C_1$ since $C_5 e^{-2x} \to 0$.

15.4 Problem 3

Consider the homogeneous differential equation

$$y^{(5)} + 4y''' = 0.$$

(a) We will once again find the general solution and then

(b) describe the asymptotic behavior of the solutions as $x \to +\infty$.

We look for solutions of the form $y = e^{\lambda x}$. The characteristic polynomial is

$$p(\lambda) = \lambda^5 + 4\lambda^3 = \lambda^3(\lambda^2 + 4) = \lambda^3(\lambda - 2i)(\lambda + 2i).$$

Hence the characteristic numbers are $\lambda = 0$ with multiplicity 3, and $\lambda = \pm 2i$. Therefore, a fundamental set of solutions is

$$\{e^{0\cdot x},\ xe^{0\cdot x},\ x^2 e^{0\cdot x},\ e^{0\cdot x}\cos(2x),\ e^{0\cdot x}\sin(2x)\} = \{1,\ x,\ x^2,\ \cos(2x),\ \sin(2x)\}$$

and the general solution is given by

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 \cos(2x) + C_5 \sin(2x)$$

for every $x \in \mathbb{R}$, where $C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}$. This gives the answer to item (a).

Now we will answer item (b). As $x \to +\infty$, the terms $\cos(2x)$ and $\sin(2x)$ remain bounded (they oscillate between -1 and 1), while the polynomial part $C_1 + C_2 x + C_3 x^2$ may grow without bound. Therefore, the long–term behavior is determined by the polynomial part, and we have the following cases.

- ⋆ If $C_3 \neq 0$, then $y(x) \sim C_3 x^2$, so $|y(x)| \to +\infty$ with quadratic growth.
- ⋆ If $C_3 = 0$ and $C_2 \neq 0$, then $y(x) \sim C_2 x$, so $|y(x)| \to +\infty$ with linear growth.
- ⋆ If $C_3 = C_2 = 0$, then $y(x) = C_1 + C_4 \cos(2x) + C_5 \sin(2x)$, which is bounded and oscillatory, so $y(x)$ does not converge unless $C_4 = C_5 = 0$ in which case $y(x) \equiv C_1$.

15.5 Problem 4

Consider the homogeneous differential equation

$$y'' - 4y' + 13y = 0.$$

(a) We will once again find the general solution and then

(b) describe the asymptotic behavior of the solutions as $x \to +\infty$.

We look for solutions of the form $y = e^{\lambda x}$. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - 4\lambda + 13.$$

Solving $p(\lambda) = 0$, we obtain

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Hence the characteristic numbers are $\lambda = 2 \pm 3i$. Therefore, a fundamental system is given by

$$\{e^{2x}\cos(3x),\ e^{2x}\sin(3x)\}$$

and the general solution is given by

$$y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$$

for every $x \in \mathbb{R}$ where $C_1, C_2 \in \mathbb{R}$. This gives the answer to item (a).

Now we will answer item (b). As $x \to +\infty$, the factor $e^{2x} \to +\infty$ while $\cos(3x)$ and $\sin(3x)$ remain bounded. Thus, every nontrivial solution grows exponentially in magnitude, while oscillating. More precisely, writing the trigonometric combination in amplitude–phase form,

$$C_1\cos(3x) + C_2\sin(3x) = R\cos(3x - \delta), \qquad R = \sqrt{C_1^2 + C_2^2},$$

we obtain

$$y(x) = R\,e^{2x}\cos(3x - \delta).$$

Therefore,

- ⋆ If $(C_1, C_2) \neq (0,0)$, then $|y(x)|$ grows like e^{2x} and the solution oscillates with increasing amplitude. In this case, there is no answer.
- ⋆ If $C_1 = C_2 = 0$, then $y(x) \equiv 0$.

15.6 Problem 5

Consider the Cauchy problem

$$\begin{cases} y'' - y' - 6y = 0, \\ y(0) = -4, \\ y'(0) = 13. \end{cases}$$

(a) We will find the general solution for the linear equation.

(b) We will then find the solution for this Cauchy problem

(c) After, we will find some initial conditions at $x_0 = 0$ so that the resulting solution satisfies $y(1) = e^3$ and $y'(1) = 3e^3$.

(d) Finally, we will developt a test that recognizes which initial conditions at $x_0 = 0$ so that the corresponding solution is bounded on $(0, +\infty)$.

We look for solutions of the form $y = e^{\lambda x}$. The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Hence the characteristic numbers are $\lambda_1 = 3$ and $\lambda_2 = -2$. Therefore, a fundamental system of solutions is $\{e^{3x}, e^{-2x}\}$ and the general solution is

$$y(x) = C_1 e^{3x} + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$. This replies item (a). Now let us consider item (b). We impose the initial conditions. First,

$$y(0) = C_1 + C_2 = -4.$$

Also,

$$y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x} \quad \Longrightarrow \quad y'(0) = 3C_1 - 2C_2 = 13.$$

Thus,

$$\begin{cases} C_1 + C_2 = -4, \\ 3C_1 - 2C_2 = 13. \end{cases}$$

From $C_2 = -4 - C_1$ and substituting into the second equation,

$$3C_1 - 2(-4 - C_1) = 13 \quad \Longrightarrow \quad 3C_1 + 8 + 2C_1 = 13 \quad \Longrightarrow \quad 5C_1 = 5 \quad \Longrightarrow \quad C_1 = 1,$$

and then $C_2 = -4 - 1 = -5$. Therefore, the solution of the Cauchy problem is

$$y(x) = e^{3x} - 5e^{-2x}$$

for every $x \in \mathbb{R}$. We will now consider item (c). From item (a), we know that the general solution of the equation is $y(x) = C_1 e^{3x} + C_2 e^{-2x}$. We want to find initial conditions at $x_0 = 0$ such that the corresponding solution satisfies $y(1) = e^3$ and $y'(1) = 3e^3$. First compute the derivative of y' as follows

$$y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x}.$$

Now impose the conditions at $x = 1$. We obtain the system

$$\begin{cases} C_1 e^3 + C_2 e^{-2} = e^3, \\ 3C_1 e^3 - 2C_2 e^{-2} = 3e^3. \end{cases}$$

Divide both equations by e^3:

$$\begin{cases} C_1 + C_2 e^{-5} = 1, \\ 3C_1 - 2C_2 e^{-5} = 3. \end{cases}$$

From the first equation, we have that $C_1 = 1 - C_2 e^{-5}$. Substitute into the second and find that

$$3(1 - C_2 e^{-5}) - 2C_2 e^{-5} = 3,$$

which gives $3 - 5C_2e^{-5} = 3$. Hence, $C_2 = 0$ and $C_1 = 1$. Therefore the solution satisfying the conditions at $x = 1$ is $y(x) = e^{3x}$. The corresponding initial conditions at $x_0 = 0$ are $y(0) = 1$ and $y'(0) = 3$. Now, let us move to item (d). From item (a) once again, the general solution is $y(x) = C_1e^{3x} + C_2e^{-2x}$. As $x \to +\infty$, the term $e^{3x} \to +\infty$, so the solution will be unbounded on $(0, +\infty)$ unless we eliminate this growing mode. Therefore, we must impose $C_1 = 0$, in which case

$$y(x) = C_2e^{-2x},$$

which is bounded (in fact, it decays to 0) on $(0, +\infty)$. At $x_0 = 0$ we have $y(0) = C_2$ and, as $y'(x) = -2C_2e^{-2x}$, we have that $y'(0) = -2C_2$. Thus, the initial conditions that produce a bounded solution are exactly those of the form $y(0) = A$ and $y'(0) = -2A$ for an arbitrary constant $A \in \mathbb{R}$.

15.7 Problem 6

It is instructive to compare the equations $y'' + ay' + by = 0$ and $y' + a(x)y = b(x)$. Both are linear differential equations, but they belong to different classes and require different solution techniques. The second equation is a first–order linear ODE. These equations can often be solved either by separation or by variation as we have seen. On the other hand, the equation $y'' + ay' + by = 0$ is a second-order linear ODE with constant coefficients. As we have seen, instead of separation or variation, the main tool is the characteristic equation. Thus, while both equations are linear, the first-order case is typically handled with variation of constants, whereas higher–order constant–coefficient equations are solved using characteristic polynomials. However, we can use sometimes the characteristic equation approach to solve a first-one linear equation as follows.

Consider the IVP

$$y' - 2y = \frac{1}{x}e^{2x}$$

with initial condition $y(-1) = 0$. We first solve the associated homogeneous equation $y' - 2y = 0$. The characteristic equation is $\lambda - 2 = 0$ and hence $\lambda = 2$. Therefore, $y_h(x) = Ce^{2x}$. Next we find a particular solution y_p by variation of constants. Since the homogeneous solution is proportional to e^{2x}, we try $y_p(x) = u(x)e^{2x}$. Then

$$y_p'(x) = u'(x)e^{2x} + 2u(x)e^{2x}.$$

Substituting into the nonhomogeneous equation,

$$\big(u'(x)e^{2x} + 2u(x)e^{2x}\big) - 2u(x)e^{2x} = \frac{1}{x}e^{2x},$$

so

$$u'(x)e^{2x} = \frac{1}{x}e^{2x} \implies u'(x) = \frac{1}{x}.$$

Integrating, we get that $u(x) = \ln|x|$. Thus, we have the following particular solution $y_p(x) = e^{2x}\ln|x|$. Therefore, the general solution is

$$y(x) = y_h(x) + y_p(x) = Ce^{2x} + e^{2x}\ln|x|$$

for every $x \neq 0$. Now apply the initial condition $y(-1) = 0$ to get

$$0 = y(-1) = e^{-2}\big(C + \ln|-1|\big) = e^{-2}(C + 0),$$

hence $C = 0$. Therefore, the solution of the IVP on $(-\infty, 0)$ is

$$y(x) = e^{2x}\ln|x|.$$