18 Practice #6: Wednesday, March 25th, 2026

18.1 Solving general linear ODEs (undetermined coefficients)

18.2 Problem 1

We solve

$$\begin{cases} y' + y = 4e^x + x, \\ y(1) = 2e - \dfrac{2}{e}. \end{cases}$$

using the characteristic polynomial for the homogeneous equation and the method of undetermined coefficients for a particular solution.

⋆ *Step 1*: Consider the homogeneous equation $y' + y = 0$. Its characteristic polynomial is $\lambda + 1 = 0$, so $\lambda = -1$. Therefore $y_h = Ce^{-x}$ is the general solution of the homogeneous equation.

⋆ *Step 2*: Write the RHS as

$$b(x) = b_1(x) + b_2(x)$$

where $b_1(x) = 4e^x$ and $b_2(x) = x$. By the superposition principle, if $y_{p,1}$ solves $y' + y = b_1(x)$ and $y_{p,2}$ solves $y' + y = b_2(x)$, then $y_p = y_{p,1} + y_{p,2}$ solves $y' + y = b(x)$.

⋆ *Step 3*: Let us find a particular solution for $b_1(x) = 4e^x$. As the exponent together with the exponential is 1 (which does not match with our characteristic number $\lambda = -1$) and there is no sines and cosines, we can try

$$y_{p,1} = Ae^x.$$

Then, $y'_{p,1} = Ae^x$, so $y'_{p,1} + y_{p,1} = 2Ae^x$. Matching coefficients with $4e^x$ gives $2A = 4$, hence $A = 2$. Thus

$$y_{p,1} = 2e^x.$$

⋆ *Step 4*: Let us find a particular solution for $b_2(x) = x$. Since the RHS is a polynomial of degree 1 (notice that, in this case, there is no exponential; well, there is, but in the form $e^{0\cdot x}$)), we can try

$$y_{p,2} = Ax + B.$$

Then, $y'_{p,2} = A$, and therefore $y'_{p,2} + y_{p,2} = A + Ax + B = Ax + (A + B)$. Matching coefficients with x gives $A = 1$ and $A + B = 0$. Hence $B = -1$ and

$$y_{p,2} = x - 1.$$

⋆ *Step 5*: Combining the particular solutions, we obtain

$$y_p = 2e^x + x - 1.$$

Thus the general solution is

$$y = y_h + y_p = Ce^{-x} + 2e^x + x - 1.$$

⋆ *Step 6*: Use the initial condition $y(1) = 2e - \frac{2}{e}$. Substituting $x = 1$ gives

$$Ce^{-1} + 2e + 1 - 1 = 2e - \frac{2}{e}.$$

Hence $\frac{C}{e} = -\frac{2}{e}$, so $C = -2$.

⋆ *Step 7*: Therefore the solution of the initial value problem is

$$y(x) = -2e^{-x} + 2e^{x} + x - 1$$

for every $x \in \mathbb{R}$.

18.3 Problem 2

We explain how to construct the guesses for particular solutions in the table using the method of undetermined coefficients. The general rule is the following: if the RHS has the form

$$b(x) = e^{\alpha x}\Big(P_n(x)\cos(\beta x) + Q_n(x)\sin(\beta x)\Big),$$

where P_n and Q_n are polynomials of degree n, then we guess a particular solution of the form

$$y_p = e^{\alpha x}\Big(\tilde{P}_n(x)\cos(\beta x) + \tilde{Q}_n(x)\sin(\beta x)\Big),$$

where $\tilde{P}_n$ and $\tilde{Q}_n$ are polynomials of the same degree with unknown coefficients. If $\alpha + i\beta$ is a root of the characteristic polynomial with multiplicity m, then the whole guess must be multiplied by x^m as a correction of the guess.

$y''' - 4y' = 0$	$y'' - 4y' + 4y = 0$	$L[y] = b(x)$
$x \cdot (Ax + B) \cdot e^{-2x}$	$(Ax + B) \cdot e^{-2x}$	$x \cdot e^{-2x}$
$A \cdot e^{2x} \cdot \cos(x) + B \cdot e^{2x} \cdot \sin(x)$	$A \cdot e^{2x} \cdot \cos(x) + B \cdot e^{2x} \cdot \sin(x)$	$3 \cdot e^{2x} \cdot \cos x$
$(Ax^2 + Bx + C) \cdot x$	$Ax^2 + Bx + C$	$5x^2 - 1$
$(Ax + B) \cdot \cos(\pi x) + (Cx + D)\sin(\pi x)$	$(Ax + B)\cos(\pi x) + (Cx + D)\sin(\pi x)$	$13 \cdot x \cdot \cos(\pi x)$
$Ae^{x} + x \cdot Be^{2x}$	$Ae^{x} + x^2 \cdot Be^{2x}$	$e^{x} - 3e^{2x}$
$x \cdot (Ax + B) + C\sin(3x) + D\cos(3x)$	$(Ax + B) + C\cos(3x) + D\sin(3x)$	$2x - \sin(3x)$

Before explaining each row of the table, notice that the homogeneous equations are the same in all cases. Indeed, the first two columns correspond to the differential operators

$$L_1[y] = y''' - 4y' \quad \text{and} \quad L_2[y] = y'' - 4y' + 4y.$$

The characteristic polynomial of L_1 is $\lambda^3 - 4\lambda = \lambda(\lambda - 2)(\lambda + 2)$, so the characteristic numbers are $\lambda = 0, -2, 2$. The characteristic polynomial of L_2 is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so the only characteristic number is $\lambda = 2$, with multiplicity 2.

⋆ **Case $b(x) = xe^{-2x}$.** The RHS has the form $e^{-2x}P_1(x)$, where $P_1(x)$ is a polynomial of degree 1. The standard guess is therefore $y_p = (Ax + B)e^{-2x}$. For the equation $y''' - 4y' = 0$, the characteristic number $\lambda = -2$ appears with multiplicity 1. Hence we multiply the guess by x and obtain

$$y_p = x(Ax + B)e^{-2x}.$$

For the equation $y'' - 4y' + 4y = 0$, the only characteristic number is $\lambda = 2$, so -2 is not a root and the standard guess remains

$$y_p = (Ax + B)e^{-2x}.$$

⋆ **Case $b(x) = 3xe^{2x}\cos x$.** This term has the form $e^{2x}\cos x$. Therefore we take the guess

$$y_p = e^{2x}\Big(A\cos x + B\sin x\Big) = Ae^{2x}\cos(x) + Be^{2x}\sin(x).$$

The corresponding complex frequencies are $\alpha + i\beta = 2 \pm i$. These are not characteristic numbers of either homogeneous equation, so no additional factor of x is needed in both cases.

⋆ **Case $b(x) = 5x^2 - 1$.** This is a polynomial of degree 2, so the standard guess is $y_p = Ax^2 + Bx + C$. For the equation $y''' - 4y' = 0$, the characteristic number $\lambda = 0$ appears with multiplicity 1. Since polynomials correspond to the exponential e^{0x}, we must multiply the guess by x. Thus

$$y_p = x \cdot (Ax^2 + Bx + C).$$

For the equation $y'' - 4y' + 4y = 0$, 0 is not a characteristic number, so the guess remains

$$y_p = Ax^2 + Bx + C.$$

⋆ **Case $b(x) = 13x\cos(\pi x)$.** This has the form $P_1(x)\cos(\pi x)$. Hence we guess

$$y_p = (Ax + B)\cos(\pi x) + (Cx + D)\sin(\pi x).$$

The corresponding complex numbers $\pm i\pi$ are not characteristic numbers of either homogeneous equation, so the guess is the same in both columns.

⋆ **Case $b(x) = e^x - 3e^{2x}$.** We treat each exponential separately. For e^x we try Ae^x. Since $\lambda = 1$ is not a characteristic number of either equation, this term does not need modification. For e^{2x} we check whether $\lambda = 2$ is a root. For $y''' - 4y' = 0$, $\lambda = 2$ is a root of multiplicity 1, so we multiply the guess by x, so

$$y_p = Ae^x + x \cdot Be^{2x}.$$

For $y'' - 4y' + 4y = 0$, $\lambda = 2$ is a root of multiplicity 2, so we multiply by x^2 and get

$$y_p = Ae^x + x^2 \cdot Be^{2x}.$$

⋆ **Case $b(x) = 2x - \sin(3x)$.** We again treat each term separately. For $2x$ we start with the polynomial guess $Ax + B$. For the equation $y''' - 4y' = 0$, $\lambda = 0$ is a characteristic number, so we multiply by x: $x(Ax + B)$. For $\sin(3x)$ we use the trigonometric guess $C\cos(3x) + D\sin(3x)$. Since $\pm 3i$ are not characteristic numbers of either equation, no extra power of x is required. Combining the terms gives

$$y_p = x \cdot (Ax + B) + C\sin(3x) + D\cos(3x)$$

for $y''' - 4y' = 0$, and

$$y_p = (Ax + B) + C\cos(3x) + D\sin(3x)$$

for $y'' - 4y' + 4y = 0$.

18.4 Problem 3

We solve

$$\begin{cases} y'' - y' = 4\sin(x) - 2x, \\ y(0) = 3, \quad y'(0) = 0. \end{cases}$$

⋆ *Step 1*: Consider the homogeneous equation $y'' - y' = 0$. Its characteristic polynomial is $\lambda^2 - \lambda = \lambda(\lambda - 1)$, so the characteristic numbers are $\lambda = 0$ and $\lambda = 1$. Therefore

$$y_h = C_1 + C_2 e^x$$

is the general solution of the homogeneous equation.

⋆ *Step 2*: Write the RHS as $b(x) = b_1(x) + b_2(x)$, where $b_1(x) = 4\sin x$ and $b_2(x) = -2x$. By the superposition principle, if $y_{p,1}$ solves $y'' - y' = b_1(x)$ and $y_{p,2}$ solves $y'' - y' = b_2(x)$, then $y_p = y_{p,1} + y_{p,2}$ solves the whole equation.

⋆ *Step 3*: Let us find a particular solution for $b_1(x) = 4\sin x$. Since the RHS has the form $\sin x$, we try

$$y_{p,1} = A\cos x + B\sin x$$

with no correction. Compute $y'_{p,1} = -A\sin x + B\cos x$ and $y''_{p,1} = -A\cos x - B\sin x$. Hence, $y''_{p,1} - y'_{p,1} = (-A - B)\cos x + (A - B)\sin x$. Matching coefficients with $4\sin x$ gives the system

$$\begin{cases} -A - B = 0, \\ A - B = 4. \end{cases}$$

From the first equation $B = -A$. Substituting into the second gives $2A = 4$, so $A = 2$ and $B = -2$. Therefore

$$y_{p,1} = 2\cos x - 2\sin x.$$

⋆ *Step 4*: Let us find a particular solution for $b_2(x) = -2x$. Since the RHS is a polynomial of degree 1, $Ax + B$. However, 0 is a root of the characteristic polynomial, so we need to add x and therefore $y_{p,2}$ should be

$$y_{p,2} = (Ax + B) \cdot x = Ax^2 + Bx.$$

Then, $y'_{p,2} = 2Ax + B$ and $y''_{p,2} = 2A$. Hence $y''_{p,2} - y'_{p,2} = 2A - (2Ax + B) = -2Ax + (2A - B)$. Matching coefficients with $-2x$ gives $-2A = -2$ and $2A - B = 0$. Thus $A = 1$ and $B = 2$. Therefore

$$y_{p,2} = x^2 + 2x.$$

⋆ *Step 5*: Combining the particular solutions we obtain

$$y_p = 2\cos x - 2\sin x + x^2 + 2x.$$

Thus the general solution is

$$y = y_h + y_p = C_1 + C_2 e^x + 2\cos x - 2\sin x + x^2 + 2x.$$

⋆ *Step 6*: Use the initial conditions. First, $y(0) = C_1 + C_2 + 2 = 3$, so $C_1 + C_2 = 1$. Next compute

$$y'(x) = C_2 e^x - 2\sin x - 2\cos x + 2x + 2.$$

Using $y'(0) = 0$ gives $C_2 - 2 + 2 = 0$, so $C_2 = 0$. Hence $C_1 = 1$.

⋆ *Step 7*: Therefore the solution of the initial value problem is

$$y(x) = 1 + 2\cos x - 2\sin x + x^2 + 2x$$

for every $x \in \mathbb{R}$.

18.5 Problem 4

We solve

$$y''' - y'' + y' - y = 10e^{2x} - x$$

and we study the rate of growth of the general solution as $x \to \infty$.

⋆ *Step 1*: Consider the homogeneous equation $y''' - y'' + y' - y = 0$. Its characteristic polynomial is $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda^2 + 1)(\lambda - 1)$. Thus the characteristic numbers are $\lambda = 1$ and $\lambda = \pm i$. Therefore the general solution of the homogeneous equation is

$$y_h = C_1 e^x + C_2 \cos x + C_3 \sin x.$$

⋆ *Step 2*: Write the RHS as $b(x) = b_1(x) + b_2(x)$ where $b_1(x) = 10e^{2x}$ and $b_2(x) = -x$. By the superposition principle, if $y_{p,1}$ solves $y''' - y'' + y' - y = b_1(x)$ and $y_{p,2}$ solves $y''' - y'' + y' - y = b_2(x)$, then $y_p = y_{p,1} + y_{p,2}$ solves the equation.

⋆ *Step 3*: Let us find a particular solution for $b_1(x) = 10e^{2x}$. Since 2 is not a characteristic number, we try $y_{p,1} = Ae^{2x}$. Then, $y'_{p,1} = 2Ae^{2x}$, $y''_{p,1} = 4Ae^{2x}$, $y'''_{p,1} = 8Ae^{2x}$. Substituting gives

$$y'''_{p,1} - y''_{p,1} + y'_{p,1} - y_{p,1} = (8A - 4A + 2A - A)e^{2x} = 5Ae^{2x}.$$

Matching coefficients with $10e^{2x}$ yields $5A = 10$, hence $A = 2$. Thus

$$y_{p,1} = 2e^{2x}.$$

⋆ *Step 4*: Let us find a particular solution for $b_2(x) = -x$. Since the RHS is a polynomial of degree 1, we try $y_{p,2} = Ax + B$. Then, $y'_{p,2} = A$, $y''_{p,2} = 0$ and $y'''_{p,2} = 0$. Substituting into the equation gives

$$y'''_{p,2} - y''_{p,2} + y'_{p,2} - y_{p,2} = A - (Ax + B) = -Ax + (A - B).$$

Matching coefficients with $-x$ gives $-A = -1$ and $A - B = 0$. Hence $A = 1$ and $B = 1$, so

$$y_{p,2} = x + 1.$$

⋆ *Step 5*: Combining the particular solutions, we obtain

$$y_p = 2e^{2x} + x + 1.$$

Therefore the general solution is

$$y = y_h + y_p = C_1 e^x + C_2 \cos x + C_3 \sin x + 2e^{2x} + x + 1.$$

⋆ *Step 6*: Let us study the rate of growth of the solution as $x \to \infty$. The terms $C_2 \cos x$ and $C_3 \sin x$ remain bounded, while $x + 1$ grows linearly. The term $C_1 e^x$ grows exponentially, and $2e^{2x}$ grows even faster. Hence the dominant term as $x \to \infty$ is $2e^{2x}$. Therefore every solution grows asymptotically like e^{2x} as $x \to \infty$, that is, $y(x) \sim 2e^{2x}$ as $x \to \infty$.

18.6 Problem 5

For each equation we find the homogeneous solution y_h and guess a particular solution y_p using the method of undetermined coefficients. From these expressions we can determine the behavior of the solutions as $x \to \infty$.

a) Consider $y'' - y = 9e^{2x}$. The homogeneous equation is $y'' - y = 0$. Its characteristic polynomial is $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, so $y_h = C_1 e^x + C_2 e^{-x}$. Since the RHS is $9e^{2x}$ and $\lambda = 2$ is not a characteristic number, the particular solution is given by $y_p = Ae^{2x}$. The general solution therefore has the form

$$y = C_1 e^x + C_2 e^{-x} + Ae^{2x}.$$

As $x \to \infty$, the dominant term is Ae^{2x}. Hence the solutions grow like e^{2x}, that is, $y(x) \sim Ae^{2x}$.

b) Consider $y'' + 3y' + 2y = \sin x + 3\cos x$. The homogeneous equation is $y'' + 3y' + 2y = 0$. Its characteristic polynomial is $\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$, so $y_h = C_1 e^{-x} + C_2 e^{-2x}$. Since the RHS contains $\sin x$ and $\cos x$ and $\pm i$ are not characteristic numbers, we consider the particular solution $y_p = A\cos x + B\sin x$. The general solution therefore has the form

$$y = C_1 e^{-x} + C_2 e^{-2x} + A\cos x + B\sin x.$$

As $x \to \infty$, the exponential terms decay to 0, while the trigonometric terms remain bounded. Hence the solutions remain bounded as $x \to \infty$ and there is no dominant terms at ∞.

c) Consider $y'' - 4y' + 4y = e^{2x}$. The homogeneous equation is $y'' - 4y' + 4y = 0$. Its characteristic polynomial is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so $y_h = C_1 e^{2x} + C_2 x e^{2x}$. Since the RHS is e^{2x} and $\lambda = 2$ is a characteristic number of multiplicity 2, we multiply the usual guess by x^2 and consider the particular solution $y_p = Ax^2 e^{2x}$. The general solution therefore has the form

$$y = C_1 e^{2x} + C_2 x e^{2x} + Ax^2 e^{2x}.$$

As $x \to \infty$, all terms behave like a polynomial times e^{2x}. Hence the solutions grow like $Ax^2 e^{2x}$, that is, $y(x) \sim Ax^2 e^{2x}$.

d) Consider $y'' - 4y = -8x$. The homogeneous equation is $y'' - 4y = 0$. Its characteristic polynomial is $\lambda^2 - 4 = (\lambda - 2)(\lambda + 2)$, so $y_h = C_1 e^{2x} + C_2 e^{-2x}$. Since the RHS is a polynomial of degree 1 and 0 is not a characteristic number, we try the particular solution $y_p = Ax + B$. The general solution therefore has the form

$$y = C_1 e^{2x} + C_2 e^{-2x} + Ax + B.$$

As $x \to \infty$, the dominant term is $C_1 e^{2x}$. Hence the solutions grow like e^{2x}, that is, $y(x) \sim C_1 e^{2x}$.

18.7 Problem 6

We solve the following Cauchy problem

$$\begin{cases} y'' + 2y' = 6e^x - 2e^{-2x} + 8\cos(2x) + 4x, \\ y(0) = 2, \\ y'(0) = 2. \end{cases}$$

⋆ *Step 1*: Consider the homogeneous equation $y'' + 2y' = 0$. Its characteristic polynomial is $\lambda^2 + 2\lambda = \lambda(\lambda + 2)$, so the characteristic numbers are $\lambda = 0$ and $\lambda = -2$. Therefore

$$y_h = C_1 + C_2 e^{-2x}$$

is the general solution of the homogeneous equation.

⋆ *Step 2*: Write the RHS as $b(x) = b_1(x) + b_2(x) + b_3(x) + b_4(x)$, where $b_1(x) = 6e^x$, $b_2(x) = -2e^{-2x}$, $b_3(x) = 8\cos(2x)$, $b_4(x) = 4x$. By the superposition principle, if $y_{p,i}$ solves $y'' + 2y' = b_i(x)$, then $y_p = y_{p,1} + y_{p,2} + y_{p,3} + y_{p,4}$ solves the whole equation.

⋆ *Step 3*: Let us find a particular solution for $b_1(x) = 6e^x$. Since $\lambda = 1$ is not a characteristic number, we try $y_{p,1} = Ae^x$. Then $y'_{p,1} = Ae^x$, $\quad y''_{p,1} = Ae^x$. Hence $y''_{p,1} + 2y'_{p,1} = 3Ae^x$. Matching coefficients with $6e^x$ gives $3A = 6$, so $A = 2$. Thus

$$y_{p,1} = 2e^x.$$

⋆ *Step 4*: Let us find a particular solution for $b_2(x) = -2e^{-2x}$. Since $\lambda = -2$ is a characteristic number of multiplicity 1, we multiply the usual guess by x and try $y_{p,2} = Axe^{-2x}$. Compute $y'_{p,2} = A(1 - 2x)e^{-2x}$ and $y''_{p,2} = A(-4 + 4x)e^{-2x}$. Hence $y''_{p,2} + 2y'_{p,2} = -2Ae^{-2x}$. Matching coefficients with $-2e^{-2x}$ gives $A = 1$. Therefore

$$y_{p,2} = xe^{-2x}.$$

⋆ *Step 5*: Let us find a particular solution for $b_3(x) = 8\cos(2x)$. Since $\pm 2i$ are not characteristic numbers, we try $y_{p,3} = A\cos(2x) + B\sin(2x)$. Then $y'_{p,3} = -2A\sin(2x) + 2B\cos(2x)$ and $y''_{p,3} = -4A\cos(2x) - 4B\sin(2x)$. Hence, $y''_{p,3} + 2y'_{p,3} = (-4A + 4B)\cos(2x) + (-4B - 4A)\sin(2x)$. Matching coefficients with $8\cos(2x)$ gives the system

$$\begin{cases} -4A + 4B = 8, \\ -4B - 4A = 0. \end{cases}$$

From the second equation $B = -A$. Substituting into the first gives $-8A = 8$, so $A = -1$ and $B = 1$. Thus

$$y_{p,3} = -\cos(2x) + \sin(2x).$$

⋆ *Step 6*: Let us find a particular solution for $b_4(x) = 4x$. Since the RHS is a polynomial of degree 1 and $\lambda = 0$ is a characteristic number, we multiply the usual guess by x and try $y_{p,4} = Ax^2 + Bx$. Then, $y'_{p,4} = 2Ax + B$, $\quad y''_{p,4} = 2A$. Hence, $y''_{p,4} + 2y'_{p,4} = 4Ax + (2A + 2B)$. Matching coefficients with $4x$ gives $A = 1$ and $A + B = 0$, so $B = -1$. Therefore

$$y_{p,4} = x^2 - x.$$

⋆ *Step 7*: Combining the particular solutions, we obtain

$$y_p = 2e^x + xe^{-2x} - \cos(2x) + \sin(2x) + x^2 - x.$$

Thus the general solution is

$$y = y_h + y_p = C_1 + C_2 e^{-2x} + 2e^x + xe^{-2x} - \cos(2x) + \sin(2x) + x^2 - x$$

for every $x \in \mathbb{R}$.

⋆ *Step 8*: Using that $y(0) = y'(0) = 1$, we can find that $C_1 = 0$ and $C_2 = 1$. Therefore,

$$y(x) = e^{-2x} + 2e^x + xe^{-2x} - \cos(2x) + \sin(2x) + x^2 - x$$

for every $x \in \mathbb{R}$.