ODE: Practice problems—Linear differential equations

1. For each of the following left-hand sides with constant coefficients

a) $y'' - 4y' + 3y$;   b) $y'' - 2y' + 5y$;   c) $y''' - 4y'' + 13y'$;   d) $y^{(4)} + 9y''$

guess using the method of undetermined coefficients the general form of a particular solution (you do not have to determine the actual coefficients) for all the following special right-hand sides:

α) $(x+1)e^{3x}$;   β) $x^2 + 1$;   γ) $12\sin(3x)$;

δ) $(x^2 - 3)e^{2x}$;   ε) $2e^x \sin(2x) + (x-1)e^x \cos(2x)$.

Find general solutions of the following equations:

2. $y''' - 2y'' = 2e^x - 1$;

3. $x'' - 3x' + 2x = \sin(t) - t\cos(t) - 1$;

4. $x'' - 3x' + 2x = 2e^t + 2t^2 - 1$;

5. $y'' + 4y = 1 + 2\sin(2x)$;

6. $y'' - 2y' = 2x - 1 + x\,e^x$;

7. $y'' - 2y' = 5\sin(x) + 10\cos(x) - 8\cos(2x)$.

Solve the following initial value problems (Cauchy problems):

8. $y'' - 2y' = 2e^{2x} - 5\cos(x) + 6$, $y(0) = 2,\ y'(0) = 2$;

9. $y'' - 7y' + 12y = e^{4x} + 12x - 19$, $y(0) = 0,\ y'(0) = 5$;

10. $y'' - 6y' + 9y = 4e^x + 9x + 12$, $y(0) = 2,\ y'(0) = -1$;

11. $y'' - 4y' + 5y = 8\sin(x) + 25x$, $y(0) = 5,\ y'(0) = 6$;

12. $y''' + y'' - 4y' - 4y = 6e^x - 4x$, $y(0) = -3,\ y'(0) = 1,\ y''(0) = -2$;

13. $x'' - 2x' = 2\sinh(2t)$, $x(0) = -\frac{1}{8},\ x'(0) = \frac{1}{4}$;

14. $y'' - 4y = 13\sin(3x) - 5\cos(x)$, $y(0) = 3,\ y'(0) = 1$;

15. $y'' + 4y = 9t\sin(t) - 5e^t$, $y(0) = -3,\ y'(0) = 1$;

16. $y'' - 3y' + 2y = 2x + (\pi^4 + 5\pi^2 + 4)\sin(\pi x)$, $y(1) = \frac{5}{2} - 3\pi + e,\ y'(1) = 1 - \pi(2 - \pi^2) + e$;

17. $\ddot{x} + x = \sin(t) + e^t \sin(t)$, $x(0) = -\frac{2}{5},\ \dot{x}(0) = \frac{13}{10}$.

Solutions

1.
a: Left-hand side $y'' - 4y' + 3y = \cdots$: Char. pol. $p(\lambda) = \lambda^2 - 4\lambda + 3$, char. numbers $\lambda = 1, 3$.
aα) $y'' - 4y' + 3y = (x+1)e^{3x}$:   Left-hand side: $\lambda = 1, 3$;
right-hand side: degree of polynomial is 1, no sines/cosines, hence $\lambda = 3 + 0i = 3$, match
with the left-hand side of multiplicity $m = 1$, thus we guess
$y_p(x) = x^1(Ax + B)e^{3 \cdot x} = (Ax^2 + Bx)e^{3x}$.
aβ) $y'' - 4y' + 3y = x^2 + 1$:   Left-hand side: $\lambda = 1, 3$;
right-hand side: degree of polynomial is 2, no exponentials nor sines/cosines, hence
$\lambda = 0 + 0i = 0$, no match with the left-hand side, thus we guess $y_p(x) = Ax^2 + Bx + C$.
aγ) $y'' - 4y' + 3y = 12\sin(3x)$:   Left-hand side: $\lambda = 1, 3$;
right-hand side: degree of polynomial is 0, no exponentials, hence $\lambda = 0 + 3i = 3i$, no match
with left-hand side, thus we guess $y_p(x) = A\sin(3x) + B\cos(3x)$.
aδ) $y'' - 4y' + 3y = (x^2 - 3)e^{2x}$:   Left-hand side: $\lambda = 1, 3$;
right-hand side: degree of polynomial is 2, no sines/cosines, hence $\lambda = 2 + 0i = 2$, no match
with left-hand side, thus we guess $y_p(x) = (Ax^2 + Bx + C)e^{2x}$.
aε) $y'' - 4y' + 3y = 2e^x \sin(2x) + (x-1)e^x \cos(2x)$:   Left-hand side: $\lambda = 1, 3$;
right-hand side: Both exponential and (co)sine, so $\lambda = 1 + 2i$, match, hence $m = 0$; max.
degree of polynomial is $d = 1$; we thus have $y_p(x) = e^x[(Ax + B)\sin(2x) + (Cx + D)\cos(2x)]$.

b: Left-hand side $y'' - 2y' + 5y = \cdots$: Char. pol. $p(\lambda) = \lambda^2 - 2\lambda + 5$, char. numbers $\lambda = 1 \pm 2i$;
bα) $y'' - 2y' + 5y = (x+1)e^{3x}$:   Left-hand side: $\lambda = 1 \pm 2i$;
right-hand side: degree of polynomial is 1; we have $\lambda = 3 + 0i = 3$, no match with left-hand
side, thus we guess $y_p(x) = (Ax + B)e^{3x}$.
bβ) $y'' - 2y' + 5y = x^2 + 1$:   Left-hand side: $\lambda = 1 \pm 2i$;
right-hand side: degree of polynomial is 2, we have $\lambda = 0 + 0i = 0$, no match with left-hand
side, thus we guess $y_p(x) = Ax^2 + Bx + C$.
bγ) $y'' - 2y' + 5y = 12\sin(3x)$:   Left-hand side: $\lambda = 1 \pm 2i$;
right-hand side: degree of polynomial is 0; we have $\lambda = 0 + 3i = 3i$, no match with left-hand
side, thus we guess $y_p(x) = A\sin(3x) + B\cos(3x)$.
bδ) $y'' - 2y' + 5y = (x^2 - 3)e^{2x}$:
Left-hand side: $\lambda = 1 \pm 2i$;
right-hand side: degree of polynomial is 2, we have $\lambda = 2 + 0i = 2$, no match with left-hand
side, thus we guess $y_p(x) = (Ax^2 + Bx + C)e^{2x}$.
bε) $y'' - 2y' + 5y = 2e^x \sin(2x) + (x-1)e^x \cos(2x)$:   Left-hand side: $\lambda = 1 \pm 2i$;
right-hand side: maximal degree of polynomial is $d = 1$; we have $\lambda = 1 + 2i$, match with left-
hand side of multiplicity $m = 1$; thus we guess $y_p(x) = x^1 e^x[(Ax + B)\sin(x) + (Cx + D)\cos(x)]$
$= e^x[(Ax^2 + Bx)\sin(2x) + (Cx^2 + Dx)\cos(2x)]$.

c: Left-hand side $y''' - 4y'' + 13y' = \cdots$: Char. pol. $p(\lambda) = \lambda^3 - 4\lambda^2 + 13\lambda$, char. numbers
$\lambda = 0, 2 \pm 3i$;
cα) $y''' - 4y'' + 13y' = (x+1)e^{3x}$:   Left-hand side: $\lambda = 0, 2 \pm 3i$;
right-hand side: degree of polynomial 1; we have $\lambda = 3 + 0i = 3$, no match with lweft-hand
side, thus we guess $y_p(x) = (Ax + B)e^{3x}$.
cβ) $y''' - 4y'' + 13y' = x^2 + 1$:   Left-hand side: $\lambda = 0, 2 \pm 3i$;
right-hand side: degree of polynomial is 2; we have $\lambda = 0 + 0i = 0$, match with left-hand side
of multiplicity 1, thus we guess $y_p(x) = x^1(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$.
cγ) $y''' - 4y'' + 13y' = 12\sin(3x)$:   Left-hand side: $\lambda = 0, 2 \pm 3i$;

right-hand side: degree of polynomial is $d = 0$; we have $\lambda = 0 + 3i = 3i$, no match with left-hand side, thus we guess $y_p(x) = A\sin(3x) + B\cos(3x)$.

cδ) $y''' - 4y'' + 13y' = (x^2 - 3)e^{2x}$:   Left-hand side: $\lambda = 0, 2 \pm 3i$;
right-hand side: degree of polynomial is 2; we have $\lambda = 2 + 0i = 2$, no match with left-hand side, thus we guess $y_p(x) = (Ax^2 + Bx + C)e^{2x}$.

cε) $y''' - 4y'' + 13y' = 2e^x\sin(2x) + (x-1)e^x\cos(2x)$:   Left-hand side: $\lambda = 0, 2 \pm 3i$;
right-hand side: max. degree of polynomial is $d = 1$, we have $\lambda = 1 + 2i$, no match with left-hand side, thus we guess $y_p(x) = e^x[(Ax + B)\sin(2x) + (Cx + D)\cos(2x)]$.

dα) Left-hand side: $y^{(4)} + 9y'' = \cdots$:
Char. pol. $p(\lambda) = \lambda^4 + 9\lambda^2$, char. numbers $\lambda = 0$ $(2\times), \pm 3i$;

dα) $y^{(4)} + 9y'' = (x+1)e^{3x}$:   Left-hand side: $\lambda = 0$ $(2\times), \pm 3i$;
right-hand side: degree of polynomial is 1; we have $\lambda = 3 + 0i = 3$, no match with left-hand side, thus we guess $y_p(x) = (Ax + B)e^{3x}$.

dβ) $y^{(4)} + 9y'' = x^2 + 1$:   Left-hand side: $\lambda = 0$ $(2\times), \pm 3i$;
right-hand side: degree of polynomial is 2; we have $\lambda = 0 + 0i = 0$, match with left-hand side of multiplicity $m = 2$; thus we guess $y_p(x) = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2$.

dγ) $y^{(4)} + 9y'' = 12\sin(3x)$:   Left-hand side: $\lambda = 0$ $(2\times), \pm 3i$;
right-hand side: degree of polynomial is 0; we have $\lambda = 0 + 3i = 3i$, match with left-hand side of multiplicity $m = 1$; thus we guess
$y_p(x) = x^1[\sin(3 \cdot x) + B\cos(3 \cdot x)] = Ax\sin(3x) + Bx\cos(3x)$.

dδ) $y^{(4)} + 9y'' = (x^2 - 3)e^{2x}$:   Left-hand side: $\lambda = 0$ $(2\times), \pm 3i$;
right-hand side: degree of polynomial is 2; we have $\lambda = 2 + 0i = 2$, no match with left-hand side, thus we guess $y_p(x) = (Ax^2 + Bx + C)e^{2x}$.

dε) $y^{(4)} + 9y'' = 2e^x\sin(2x) + (x-1)e^x\cos(2x)$:   Left-hand side: $\lambda = 0$ $(2\times), \pm 3i$;
right-hand side: max. degree of polynomial is $d = 1$; we have $\alpha = 1 + 2i$, no match with left-hand side, thus we guess $y_p(x) = e^x[(Ax + B)\sin(2x) + (Cx + D)\cos(2x)]$.

2. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^3 - 2\lambda^2$, char. numbers $\lambda = 0$ $(2\times), 2$; fund. syst. $\{1, x, e^{2x}\}$; general solution of homogeneous equation is $y_h(x) = a + bx + c\,e^{2x}$, $x \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
• $2e^x$: $d = 0$; $\lambda = 1 + 0i = 1$, no match, hence $y_1(x) = A\,e^x$.
• -1: $d = 0$; $\lambda = 0 + 0i = 0$, match with left-hand side of multiplicity $m = 2$; hence $y_2(x) = x^2 C = Cx^2$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = A\,e^x + Cx^2$, substitute into the given equation and obtain
$$-A\,e^x - 4C = 2e^x - 1, \text{ so } A = -2, \ C = \tfrac{1}{4},$$
general solution is $y(x) = y_p(x) + y_h(x) = \frac{1}{4}x^2 - 2e^x + a + bx + c\,e^{2x}$, $x \in \mathbb{R}$.

3. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 3\lambda + 2$, char. numbers $\lambda = 1, 2$; fund. syst. $\{e^t, e^{2t}\}$; general solution of homogeneous equation is $x_h(t) = a\,e^t + b\,e^{2t}$, $t \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
• $\sin(t) - t\cos(t)$: $d = 1$; $\lambda = 0 + 1i = i$, no match with left-hand side, hence $x_1(t) = (At + B)\sin(t) + (Ct + D)\cos(t)$.
• -1: $d = 0$; $\lambda = 0 + 0i = 0$, no match with left-hand side, hence $x_2(t) = E$.
Guess particular solution of the form
$x_p(t) = x_1(t) + x_2(t) = (At + B)\sin(t) + (Ct + D)\cos(t) + E$, substitute into the given equation and obtain
$[(B - 3A + 3D - 2C) + (A + 3C)t]\sin(t) + [(D - 3B - 3C + 2A) + (C - 3A)t]\cos(t) + 2E$

$$= \sin(t) - t\cos(t) - 1,$$

hence $B - 3A + 3D - 2C = 1$, $A + 3C = 0$, $D - 3B - 3C + 2A = 0$, $C - 3A = -1$, $2E = -1$,
so $A = \frac{3}{10}$, $B = \frac{11}{25}$, $C = \frac{-1}{10}$, $D = \frac{21}{50}$, $E = -\frac{1}{2}$, general solution is
$x(t) = x_p(t) + x_h(t) = \left(\frac{3}{10}t + \frac{11}{25}\right)\sin(t) + \left(\frac{21}{50} - \frac{1}{10}t\right)\cos(t) - \frac{1}{2} + a\,e^t + b\,e^{2t}$, $t \in \mathbb{R}$.

4. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 3\lambda + 2$, char.
numbers $\lambda = 1, 2$; fund. syst. $\{e^t, e^{2t}\}$; general solution of homogeneous equation is
$x_h(t) = a\,e^t + b\,e^{2t}$, $t \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
● $2e^t$: $d = 0$; $\lambda = 1 + 0i = 1$, match with left-hand side of multiplicity 1, hence
$x_1(t) = t[A\,e^t] = At\,e^t$.
● $2t^2 - 1$: $d = 2$; $\lambda = 0 + 0i = 0$, no match, hence $x_2(t) = Ct^2 + Dt + E$.
Guess particular solution of the form $x_p(t) = x_1(t) + x_2(t) = Ate^t + Ct^2 + Dt + E$, substitute
into the given equation and obtain
$$-Ae^t + (2C - 3D + 2E) + (2D - 6C)t + 2Ct^2 = 2e^t + 2t^2 - 1,$$
so $A = -2$, $2C - 3D + 2E = -1$, $2D - 6C = 0$, $C = 1$, hence $E = 3$, $D = 3$, general solution
is $x(t) = x_p(t) + x_h(t) = -2te^t + t^2 + 3t + 3 + a\,e^t + b\,e^{2t}$, $t \in \mathbb{R}$.

5. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 + 4$, char. numbers
$\lambda = \pm 2i$; fund. syst. $\{\sin(2x), \cos(2x)\}$; general solution of homogeneous equation is
$y_h(x) = a\sin(2x) + b\cos(2x)$, $x \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
● 1: $d = 0$; $\lambda = 0 + 0i = 0$, no match with left-hand side, hence $y_1(x) = A$.
● $2\sin(2x)$: $d = 0$; $\lambda = 0 + 2i = 2i$, single match with left-hand side, hence
$y_2(x) = x[C\sin(2x) + D\cos(2x)]$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = A + Cx\sin(2x) + Dx\cos(2x)$,
substitute into the given equation and obtain
$$4A + (-4D)\sin(2x) + (4C)\cos(2x) = 1 + 2\sin(2x), \text{ so } A = \tfrac{1}{4}, C = 0, D = -\tfrac{1}{2},$$
general solution is $y(x) = y_p(x) + y_h(x) = \frac{1}{4} - \frac{1}{2}x\cos(2x) + a\sin(2x) + b\cos(2x)$, $x \in \mathbb{R}$.

6. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 2\lambda$, char. numbers
$\lambda = 0, 2$; fund. syst. $\{1, e^{2x}\}$; general solution of homogeneous equation is
$y_h(x) = a + b\,e^{2x}$, $x \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
● $2x - 1$: $d = 1$; $\lambda = 0 + 0i = 0$, single match with left-hand side, hence
$y_1(x) = x(Ax + B) = Ax^2 + Bx$.
● $x\,e^x$: $d = 1$; $\lambda = 1 + 0i = 1$, no match with left-hand side, hence $y_2(x) = (Cx + D)e^x$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = Ax^2 + Bx + (Cx + D)e^x$,
substitute into the given equation and obtain
$$[(2A - 2B) - 4Ax] + [-Cx - D]e^x = 2x - 1 + x\,e^x,$$
so $2A - 2B = -1$, $A = -\frac{1}{2}$, $C = -1$, $D = 0$, hence $B = 0$, general solution is
$y(x) = y_p(x) + y_h(x) = -\frac{1}{2}x^2 - x\,e^x + a + b\,e^{2x}$, $x \in \mathbb{R}$.

7. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 2\lambda$, char. numbers
$\lambda = 0, 2$; fund. syst. $\{1, e^{2x}\}$; general solution of homogeneous equation is
$y_h(x) = a + b\,e^{2x}$, $x \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
● $5\sin(x) + 10\cos(x)$: $d = 0$; $\lambda = 0 + 1i = i$, no match with left-hand side, hence
$y_1(x) = A\sin(x) + B\cos(x)$.
● $-8\cos(2x)$: $d = 0$; $\lambda = 0 + 2i = 2i$, no match with left-hand side, hence
$y_2(x) = C\sin(2x) + D\cos(2x)$.
Guess particular solution of the form

$y_p(x) = y_1(x) + y_2(x) = A\sin(x) + B\cos(x) + C\sin(2x) + D\cos(2x)$, substitute into the given equation and obtain
$[2B - A]\sin(x) + [-2A - B]\cos(x) + [4D - 4C]\sin(2x) + [-4C - 4D]\cos(2x)$
$\qquad = 5\sin(x) + 10\cos(x) - 8\cos(2x)$,
hence $2B - A = 5$, $-2A - B = 10$, $4D - 4C = 0$, $-4C - 4D = -8$, so $A = -5$, $B = 0$, $C = 1$, $D = 1$, general solution is $y(x) = -5\sin(x) + \sin(2x) + \cos(2x) + a + b\,e^{2x}$, $x \in \mathbb{R}$.

8. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 2\lambda$, char. numbers $\lambda = 0, 2$; fund. syst. $\{1, e^{2x}\}$; general solution of homogeneous equation is
$y_h(x) = a + b\,e^{2x}$, $x \in \mathbb{R}$.
Right-hand side is a combination of three special right-hand sides.
• $2e^{2x}$: $d = 0$; $\lambda = 2$, match of multiplicity $m = 1$ with left-hand side, hence correction, $y_1(x) = Ax\,e^{2x}$.
• $-5\cos(x)$: $d = 0$; $\lambda = i$, no match with left-hand side, hence $y_2(x) = B\cos(x) + C\sin(x)$.
• 6: $d = 0$; $\lambda = 0$, match of multiplicity $m = 1$ with left-hand side, hence correction, $y_3(x) = Dx$.
Guess particular solution of the form
$y_p(x) = y_1(x) + y_2(x) + y_3(x) = Ax\,e^{2x} + B\cos(x) + C\sin(x) + Dx$,
substitute into the given equation and obtain
$\qquad 2A\,e^{2x} + [-B - 2C]\cos(x) + [2B - C]\sin(x) - 2D = 2e^{2x} - 5\cos(x) + 6$,
hence $2A = 2$, $-B - 2C = -5$, $2B - C = 0$, $-2D = 6$, so $A = 1$, $B = 1$, $C = 2$, $D = -3$, general solution is
$y(x) = y_p(x) + y_h(x) = x\,e^{2x} + \cos(x) + 2\sin(x) - 3x + a + b\,e^{2x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = x\,e^{2x} + \cos(x) + 2\sin(x) - 3x + e^{2x}$, $x \in \mathbb{R}$.

9. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 7\lambda + 12$, char. numbers $\lambda = 3, 4$; fund. syst. $\{e^{3x}, e^{4x}\}$; general solution of homogeneous equation is
$y_h(x) = a\,e^{3x} + b\,e^{4x}$, $x \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• e^{4x}: $d = 0$; $\lambda = 4$, match of multiplicity $m = 1$ with left-hand side, hence correction, $y_1(x) = Ax\,e^{4x}$.
• $12x - 19$: $d = 1$; $\lambda = 0$, no match with left-hand side, hence $y_2(x) = Bx + C$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = Ax\,e^{4x} + Bx + C$, substitute into the given equation and obtain
$\qquad A\,e^{4x} + 12Bx + [-7B + 12C] = e^{4x} + 12x - 19$,
tedy $A = 1$, $12B = 12$, $-7B + 12C = -19$, so $B = 1$, $C = -1$, general solution is
$y(x) = y_p(x) + y_h(x) = x\,e^{4x} + x - 1 + a\,e^{3x} + b\,e^{4x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = x\,e^{4x} + x - 1 + e^{3x}$, $x \in \mathbb{R}$.

10. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$, char. numbers $\lambda = 3$ $(2\times)$; fund. syst. $\{e^{3x}, x\,e^{3x}\}$; general solution of homogeneous equation is $y_h(x) = a\,e^{3x} + bx\,e^{3x}$, $x \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $4e^x$: $d = 0$; $\lambda = 1$, no match with left-hand side, hence $y_1(x) = A\,e^x$.
• $9x + 12$: $d = 1$; $\lambda = 0$, no match with left-hand side, hence $y_2(x) = Bx + C$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = A\,e^x + Bx + C$, substitute into the given equation and obtain
$\qquad 4A\,e^x + 9Bx + [-6B + 9C] = 4e^x + 9x + 12$,
hence $4A = 4$, $9B = 9$, $-6B + 9C = 12$, so $A = 1$, $B = 1$, $C = 2$, general solution is
$y(x) = y_p(x) + y_h(x) = e^x + x + 2 + a\,e^{3x} + bx\,e^{3x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = x + 2 + e^x - e^{3x}$, $x \in \mathbb{R}$.

11. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 4\lambda + 5$, char. numbers $\lambda = 2 \pm i$; fund. syst. $\{e^{2x}\cos(x), e^{2x}\sin(x)\}$; general solution of homogeneous equation is $y_h(x) = a\,e^{2x}\cos(x) + b\,e^{2x}\sin(x)$, $x \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $8\sin(x)$: $d = 0$; $\lambda = i$, no match with left-hand side, hence $y_1(x) = A\cos(x) + B\sin(x)$.
• $25x$: $d = 1$; $\lambda = 0$, no match with left-hand side, hence $y_3(x) = Cx + D$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = A\cos(x) + B\sin(x) + Cx + D$, substitute into the given equation and obtain
$$[4A + 4B]\cos(x) + [-4A + 4B]\cos(x) + 5Cx + [-4C + 5D] = 8\sin(x) + 25x + 0,$$
hence $4A + 4B = 8$, $-4A + 4B = 0$, $5C = 25$, $-4C + 5D = 0$, so $A = 1$, $B = 1$, $C = 5$, $D = 4$, general solution is
$y(x) = \cos(x) + \sin(x) + 5x + 4 + a\,e^{2x}\cos(x) + b\,e^{2x}\sin(x)$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = \cos(x) + \sin(x) + 5x + 4$, $x \in \mathbb{R}$.

12. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^3 + \lambda^2 - 4\lambda - 4 = (\lambda+1)(\lambda-2)(\lambda+2)$, we guess one root, say, $\lambda = -1$, then by division $p(\lambda) = (\lambda+1)(\lambda^2 - 4)$. char. numbers $\lambda = -1, 2, -2$; fund. syst. $\{e^{-x}, e^{2x}, e^{-2x}\}$; general solution of homogeneous equation is $y_h(x) = a\,e^{-x} + b\,e^{2x} + b\,e^{-2x}$, $x \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $6e^x$: $d = 0$; $\lambda = 1$, no match with left-hand side, hence $y_1(x) = A\,e^x$.
• $-4x$: $d = 1$; $\lambda = 0$, no match with left-hand side, hence $y_2(x) = Bx + C$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = A\,e^x + Bx + C$, substitute into the given equation and obtain
$$-6A\,e^x - 4Bx + [-4B - 4C] = 6e^x - 4x + 0,$$
hence $-6A = 6$, $-4B = -4$, $-4B - 4C = 0$, so $A = -1$, $B = 1$, $C = -1$, general solution is
$y(x) = y_p(x) + y_h(x) = -e^x + x - 1 + a\,e^{-x} + b\,e^{2x} + c\,e^{-2x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = y_p(x) + y_h(x) = x - 1 - e^x - e^{-x}$, $x \in \mathbb{R}$.

13. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 2\lambda$, char. numbers $\lambda = 0, 2$; fund. syst. $\{1, e^{2t}\}$; general solution of homogeneous equation is
$x_h(t) = a + b\,e^{2t}$, $t \in \mathbb{R}$.
The given right-hand side is not special, but rewritten as $2\sinh(2t) = e^{2t} - e^{-2t}$ it is, in fact it is a combination of two special right-hand sides.
• e^{2t}: $d = 0$; $\lambda = 2$, single match with left-hand side, hence $x_1(t) = t^1[A\,e^{2t}] = At\,e^{2t}$.
• e^{-2t}: $d = 0$; $\lambda = -2$, no match with left-hand side, hence $x_2(t) = B\,e^{-2t}$.
Guess particular solution of the form $x_p(t) = x_1(t) + x_2(t) = At\,e^{2t} + B\,e^{-2t}$, substitute into the given equation and obtain
$$2A\,e^{2t} + 8B\,e^{-2t} = e^{2t} - e^{-2t}, \text{ so } A = \tfrac{1}{2}, B = -\tfrac{1}{8},$$
general solution is $x(t) = x_p(t) + x_h(t) = \tfrac{1}{2}t\,e^{2t} - \tfrac{1}{8}e^{-2t} + a + b\,e^{2t}$, $t \in \mathbb{R}$.
Init. conditions: $x(t) = \tfrac{1}{2}t\,e^{2t} - \tfrac{1}{8}e^{-2t} - \tfrac{1}{2}$, $t \in \mathbb{R}$.

14. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 4$, char. numbers $\lambda = \pm 2$; fund. syst. $\{e^{-2x}, e^{2x}\}$; general solution of homogeneous equation is
$y_h(x) = a\,e^{2x} + b\,e^{-2x}$, $x \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $13\sin(3x)$: $d = 0$; $\lambda = 3i$, no match with left-hand side, hence
$y_1(x) = A\sin(3x) + B\cos(3x)$.
• $-5\cos(x)$: $d = 0$; $\lambda = i$, no match with left-hand side, hence $y_2(x) = C\sin(x) + D\cos(x)$.
Guess particular solution of the form
$y_p(x) = y_1(x) + y_2(x) = A\sin(3x) + B\cos(3x) + C\sin(x) + D\cos(x)$, substitute into the given equation and obtain
$(-13A)\sin(3x) + (-13B)\cos(3x) + (-5C)\sin(x) + (-5D)\cos(x) = 13\sin(3x) - 5\cos(x)$,
so $A = -1$, $B = 0$, $C = 1$, $D = 0$, general solution is

$y(x) = \cos(x) - \sin(3x) + a\, e^{2x} + b\, e^{-2x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = \cos(x) - \sin(3x) + 2e^{2x}$, $x \in \mathbb{R}$.

15. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 + 4$, char. numbers $\lambda = \pm 2i$; fund. syst. $\{\sin(2t), \cos(2t)\}$; general solution of homogeneous equation is
$y_h(t) = a\sin(2t) + b\cos(2t)$, $t \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $9t\sin(t)$: $d = 1$; $\lambda = i$, no match with left-hand side, hence
$y_1(t) = (At + B)\sin(t) + (Ct + D)\cos(t)$.
• $-5e^t$: $d = 0$; $\lambda = 1$, no match with left-hand side, hence $y_2(t) = E\, e^t$.
Guess particular solution of the form
$y_p(t) = y_1(t) + y_2(t) = (At + B)\sin(t) + (Ct + D)\cos(t) + E\, e^t$, substitute into the given equation and obtain
$$[3B - 2C + 3At]\sin(t) + [3D + 2A + 3Ct]\cos(t) + 5Et^t = 9t\sin(t) - 5e^t,$$
hence $3B - 2C = 0$, $3A = 9$, $3D + 2A = 0$, $3C = 0$, $5E = -5$, so $A = 3$, $B = 0$, $C = 0$, $D = -2$, $E = -1$, general solution is
$y(t) = y_p(t) + y_h(t) = 3t\sin(t) - 2\cos(t) - e^t + a\sin(2t) + b\cos(2t)$, $t \in \mathbb{R}$.
Init. conditions: $y(t) = 3t\sin(t) - 2\cos(t) - e^t + \sin(2t)$, $t \in \mathbb{R}$.

16. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 - 3\lambda + 2$, char. numbers $\lambda = 1, 2$; fund. syst. $\{e^x, e^{2x}\}$; general solution of homogeneous equation is
$y_h(x) = a\, e^x + b\, e^{2x}$, $x \in \mathbb{R}$.
Right-hand side is special, in fact it is a combination of two special right-hand sides.
• $2x$: $d = 1$; $\lambda = 0$, no match with left-hand side, hence $y_1(x) = Ax + B$.
• $(\pi^4 + 5\pi^2 + 4)\sin(\pi x)$: $d = 0$; $\lambda = \pi i$, no match with left-hand side, hence
$y_2(x) = C\sin(\pi x) + D\cos(\pi x)$.
Guess particular solution of the form $y_p(x) = y_1(x) + y_2(x) = Ax + B + C\sin(\pi x) + D\cos(\pi x)$, substitute into the given equation and obtain
$[(2B - 3A) + 2Ax] + [2C - \pi^2 C + 3\pi D]\sin(x) + [2D - \pi^2 D - 3\pi C]\cos(x)$
$= 2x + (\pi^4 + 5\pi^2 + 4)\sin(\pi x)$,
hence $2B - 3A = 0$, $2A = 2$, $(2 - \pi^2)C + 3\pi D = (\pi^4 + 5\pi^2 + 4)$, $(2 - \pi^2)D - 3\pi C = 0$, so $A = 1$, $B = \frac{3}{2}$, $C = 2 - \pi^2$, $D = 3\pi$, general solution is
$y(x) = y_p(x) + y_h(x) = \frac{3}{2} + x + (2 - \pi^2)\sin(\pi x) + 3\pi\cos(\pi x) + a\, e^x + b\, e^{2x}$, $x \in \mathbb{R}$.
Init. conditions: $y(x) = \frac{3}{2} + x + (2 - \pi^2)\sin(\pi x) + 3\pi\cos(\pi x) + e^x$, $x \in \mathbb{R}$.

17. Left-hand side is linear with constant coefficients. Char. pol. $p(\lambda) = \lambda^2 + 1$, char. numbers $\lambda = \pm i$; fund. syst. $\{\sin(t), \cos(t)\}$; general solution of homogeneous equation is
$x_h(t) = a\sin(t) + b\cos(t)$, $t \in \mathbb{R}$.
Right-hand side is a combination of two special right-hand sides.
• $\sin(t)$: $d = 0$; $\lambda = i$, match of multiplicity $m = 1$ with left-hand side, hence
$x_1(t) = t[A\sin(t) + B\cos(t)]$.
• $e^t\sin(t)$: $d = 0$; $\lambda = 1 + i$, no match with left-hand side, hence
$x_2(t) = e^t[C\sin(t) + D\cos(t)]$.
Guess particular solution of the form
$x_p(t) = x_1(t) + x_2(t) = At\sin(t) + Bt\cos(t) + C\, e^t\sin(t) + D\, e^t\cos(t)$, substitute into the given equation and obtain
$(-2B)\sin(t) + (2A)\cos(t) + (C - 2D)e^t\sin(t) + (2C + D)e^t\cos(t) = \sin(t) + e^t\sin(t)$,
so $A = 0$, $B = -\frac{1}{2}$, $C - 2D = 1$, $2C + D = 0$, hence $C = \frac{1}{5}$, $D = -\frac{2}{5}$, general solution is
$x(t) = x_p(t) + x_h(t) = \frac{e^t}{5}\sin(t) - \frac{2e^t}{5}\cos(t) - \frac{t}{2}\cos(t) + a\sin(t) + b\cos(t)$, $t \in \mathbb{R}$.
Init. conditions: $x(t) = \frac{e^t}{5}\sin(t) - \frac{2e^t}{5}\cos(t) - \frac{t}{2}\cos(t) + 2\sin(t)$, $t \in \mathbb{R}$.