25 Lecture #17: Tuesday, April 14th, 2026

25.1 Systems of linear ODEs of order 1

A natural next step after studying single first-order differential equations is to consider several unknown functions at the same time, each one related not only to the independent variable but also to the others. This leads to the notion of a system of first-order differential equations. Such systems appear naturally in many applications, since several quantities often evolve simultaneously and influence one another. In particular, linear systems with constant coefficients form one of the most important classes, both because they arise frequently in practice and because they can be studied with a clear and effective theory.

25.2 Introduction and general theory

We start with the following definition.

Definition 25.1. By a **system of linear ODEs of order 1 with constant coefficients** we mean a system of the form

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x), \\ y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x), \\ \quad \vdots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x) \end{cases}$$

where $b_i(x)$ are right hand-sides, $a_{ij} \in \mathbb{R}$ are coefficients.

An Initial Value Problem (IVP) or Cauchy problem for such a system has initial conditions

$$y_1(x_0) = y_{1,0},\ y_2(x_0) = y_{2,0},\ \ldots,\ y_n(x_0) = y_{n,0}.$$

The system is called **homogeneous** if $b_i(x) = 0$ for all $i = 1, \ldots, n$.

The following theorem is the natural extension, to systems, of the existence and uniqueness result already known for a single first-order differential equation (see Theorem 13.3). It tells us that, as long as the nonhomogeneous terms are continuous, prescribing initial values for all the unknown functions at a point x_0 determines one and only one solution on the interval under consideration.

Theorem 25.2 (on existence and uniqueness for systems)**.** Consider a system of linear ODEs of order 1. If $b_i(x)$ are continuous on an open interval I, then for every $x_0 \in I$ and all $y_{1,0}, y_{2,0}, \ldots, y_{n,0} \in \mathbb{R}$ there exists a solution of the corresponding IVP on I, and it is unique.

The question now is how to solve a system of first-order linear ODEs. In what follows, we present a first approach based on a method that is already very familiar to us from elementary mathematics.

Example 25.3. Let us consider the system

$$\begin{cases} y_1' = 2y_1 + y_2 - 3, \\ y_2' = y_1 + 2y_2 + 3x - 4, \end{cases}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 1$. We solve the system by elimination (or reduction). From the first equation, we isolate y_2:

$$y_2 = y_1' - 2y_1 + 3.$$

Differentiating this expression, we obtain

$$y_2' = y_1'' - 2y_1'.$$

On the other hand, from the second equation we also have

$$y_2' = y_1 + 2y_2 + 3x - 4.$$

Substituting the expression for y_2 into this equation gives

$$y_1'' - 2y_1' = y_1 + 2(y_1' - 2y_1 + 3) + 3x - 4.$$

Expanding and simplifying, we get

$$y_1'' - 2y_1' = y_1 + 2y_1' - 4y_1 + 6 + 3x - 4,$$

and therefore

$$y_1'' - 4y_1' + 3y_1 = 3x + 2.$$

We now solve this second-order linear differential equation. First, let us consider the associated homogeneous equation

$$y_1'' - 4y_1' + 3y_1 = 0.$$

Its characteristic polynomial is $\lambda^2 - 4\lambda + 3 = 0$. Factoring, we obtain $(\lambda - 1)(\lambda - 3) = 0$, so the roots are $\lambda = 1$ and $\lambda = 3$. Hence, the general solution of the homogeneous equation is

$$y_{1,h}(x) = C_1 e^{x} + C_2 e^{3x}.$$

Next, we look for a particular solution of

$$y_1'' - 4y_1' + 3y_1 = 3x + 2$$

by the guessing method. Since the right-hand side is a polynomial of degree 1, we try

$$y_{1,p}(x) = Ax + B.$$

Then $y_{1,p}' = A$ and $y_{1,p}'' = 0$. Substituting into the equation, we get $A = 1$ and $B = 2$. Therefore, $y_{1,p}(x) = x + 2$. We conclude that the general solution for y_1 is

$$y_1(x) = C_1 e^{x} + C_2 e^{3x} + x + 2.$$

To find y_2, we use the relation $y_2 = y_1' - 2y_1 + 3$. Since $y_1'(x) = C_1 e^{x} + 3C_2 e^{3x} + 1$, we obtain

$$y_2(x) = \Big(C_1 e^{x} + 3C_2 e^{3x} + 1\Big) - 2\Big(C_1 e^{x} + C_2 e^{3x} + x + 2\Big) + 3.$$

Simplifying, we get

$$y_2(x) = -C_1 e^{x} + C_2 e^{3x} - 2x.$$

Therefore, the general solution of the system is

$$y_1(x) = C_1 e^{x} + C_2 e^{3x} + x + 2$$

and

$$y_2(x) = -C_1e^x + C_2e^{3x} - 2x.$$

We now use the initial conditions. Since $y_1(0) = 3$, we have $C_1 + C_2 + 2 = 3$, that is, $C_1 + C_2 = 1$. Since $y_2(0) = 1$, we obtain $-C_1 + C_2 = 1$. Solving this system, we find $C_1 = 0$ and $C_2 = 1$. Hence, the solution of the initial value problem is

$$y_1(x) = e^{3x} + x + 2 \quad \text{and} \quad y_2(x) = e^{3x} - 2x$$

for every $x \in \mathbb{R}$.

At first sight, one might hope that this method always works as we have the following result.

Fact 25.4. Every system of n linear ODEs of order 1 can be transformed, by elimination, into a single linear ODE of order n.

However, this is unfortunately not always manageable as the next example shows.

Example 25.5. Let us now consider the system

$$\begin{cases} y_1' = y_1 + 2y_2 + y_3 + 1, \\ y_2' = -y_1 + 2y_2 + 2y_3, \\ y_3' = 2y_1 + y_2 + y_3. \end{cases}$$

Let us try to solve this system by elimination, as in the previous example. From the first equation, we isolate y_3 as follows

$$y_3 = y_1' - y_1 - 2y_2 - 1.$$

Replacing y_3 in the second equation, we obtain

$$y_2' = -y_1 + 2y_2 + 2(y_1' - y_1 - 2y_2 - 1).$$

Expanding and simplifying, this gives

$$y_2' = 2y_1' - 3y_1 - 2y_2 - 2.$$

Rearranging, we get

$$2y_1' - y_2' = 3y_1 + 2y_2 + 2.$$

Next, we differentiate the first equation: $y_1'' = y_1' + 2y_2' + y_3'$. Using now the third equation, namely $y_3' = 2y_1 + y_2 + y_3$ and substituting again the expression for y_3, we obtain

$$y_1'' = y_1' + 2y_2' + 2y_1 + y_2 + (y_1' - y_1 - 2y_2 - 1).$$

Simplifying, this becomes

$$y_1'' = 2y_1' + 2y_2' + y_1 - y_2 - 1.$$

Hence,

$$y_1'' - 2y_1' - 2y_2' = y_1 - y_2 - 1.$$

Therefore, after elimination, we arrive at the system

$$\begin{cases} \qquad 2y_1' - y_2' = 3y_1 + 2y_2 + 2, \\ y_1'' - 2y_1' - 2y_2' = y_1 - y_2 - 1. \end{cases}$$

Thus, instead of reducing the original system to a single differential equation involving only one unknown function, elimination leads us to a new 2×2 system involving y_1 and y_2. In particular, the term y_2' still appears in the second equation, so the system does not decouple into one single ODE for y_1. This shows that, unlike in the previous example, sometimes we do not know how to use elimination to reduce a system of first-order linear equations to a single linear ODE of higher order although theoretically this should be true. However, we have the following result.

Fact 25.6. Every linear ODE of order n (and every system of linear ODEs with sum of orders n) can be equivalently transformed into a system of n linear ODEs of order 1.

Let us now consider the differential equation

$$y''' + \sin(x)\, y'' - xy' + e^x y = \ln(x)$$

with initial conditions $y(0) = 1$, $y'(0) = 13$ and $y''(0) = -1$. We introduce the new variables

$$y_1 = y, \qquad y_2 = y_1' = y', \qquad y_3 = y_2' = y''.$$

Moreover, since $y''' = y_3'$, the original equation becomes

$$y_3' + \sin(x)\, y_3 - xy_2 + e^x y_1 = \ln(x).$$

Solving for y_3', we obtain

$$y_3' = \ln(x) - \sin(x)\, y_3 + xy_2 - e^x y_1.$$

Therefore, the equation can be transformed into the following 3×3 system:

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = -e^x y_1 + xy_2 - \sin(x)\, y_3 + \ln(x). \end{cases}$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 13$ and $y_3(0) = -1$.

From a computational point of view, this reformulation is very convenient, since transforming a higher-order differential equation into a first-order system makes it much easier to implement in practice. Indeed, most numerical methods and software packages are designed precisely for systems of first-order equations. Once the new variables are introduced, the problem can be handled in a systematic way, and standard algorithms can be applied directly without any essential modification.

25.3 Matrix setup for systems

Let us now write the system in matrix form. We start with

$$\begin{cases} y_1' = 2y_1 + y_2 - 3, \\ y_2' = y_1 + 2y_2 + 3x - 4, \end{cases}$$

together with the initial conditions $y_1(0) = 3$ and $y_2(0) = 1$. We introduce the vector notation

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then its derivative is

$$\vec{y}' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}.$$

Now notice that the terms involving y_1 and y_2 can be written as a matrix multiplying the vector $\vec{y}$. Indeed,

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + y_2 \\ y_1 + 2y_2 \end{pmatrix}.$$

This shows that the coefficient matrix of the system is

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

On the other hand, the remaining terms, namely -3 and $3x - 4$, form the non-homogeneous part of the system. We write them as

$$\vec{b}(x) = \begin{pmatrix} -3 \\ 3x - 4 \end{pmatrix}.$$

Therefore, the system can be rewritten as

$$\vec{y}' = A\vec{y} + \vec{b}(x).$$

Finally, the initial conditions are also written in vector form. We define

$$\vec{y}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

so that

$$\vec{y}(0) = \vec{y}_0.$$

Hence, the whole initial value problem takes the matrix form

$$\begin{cases} \vec{y}' = A\vec{y} + \vec{b}(x), \\ \vec{y}(0) = \vec{y}_0, \end{cases}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \vec{b}(x) = \begin{pmatrix} -3 \\ 3x - 4 \end{pmatrix}, \quad \text{and} \quad \vec{y}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

More in general, we have the following definition.

Definition 25.7. A system

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + b_1(x), \\ y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + b_2(x), \\ \quad \vdots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + b_n(x) \end{cases}$$

can be written as $\vec{y}' = A\vec{y} + \vec{b}(x)$, where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

is the **matrix of the system**,

$$\vec{b}(x) = \begin{pmatrix} b_1(x) \\ \vdots \\ b_n(x) \end{pmatrix}$$

is the vector of right-hand sides, and

$$\vec{y}(x) = \begin{pmatrix} y_1(x) \\ \vdots \\ y_n(x) \end{pmatrix}$$

is the unknown vector. Then

$$\vec{y}\,' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}.$$

The system is **homogeneous** if $\vec{b} = \vec{0}$, where

$$\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}.$$

Initial conditions are written as $\vec{y}(x_0) = \vec{y}_0$.

Thanks to this notation and the whole theory behind vector functions, we have the following structural theorem. The following is the analogous of Theorem 13.4.

Theorem 25.8 (on structure of solution set for homogeneous systems)**.** Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in \mathbb{R}^{n\times n}$. The set of all solutions of this system on some open interval I is a linear space of dimension n.

We also have the analogous of Theorem 13.5.

Theorem 25.9 (on structure of solution set for systems)**.** Consider a system of linear ODEs

$$\vec{y}\,' = A\vec{y} + \vec{b}(x).$$

Let $\vec{y}_p$ be some solution of this system on I. Then $\vec{y}_0$ is another solution of this system on I if and only if

$$\vec{y}_0 = \vec{y}_p + \vec{y}_h$$

for some solution $\vec{y}_h$ of the homogeneous system $\vec{y}\,' = A\vec{y}$ on I. Thus, if $\vec{y}_h$ is a general solution of the associated homogeneous system on I, then $\vec{y}_p + \vec{y}_h$ is a general solution of the given system on I.

In order to illustrate Theorem 25.9, the vector form of the solutions we have found in Example 25.3 is given in the following way. The solution can be written as

$$\vec{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} x + 2 + C_1e^{x} + C_2e^{3x} \\ -2x - C_1e^{x} + C_2e^{3x} \end{pmatrix}.$$

We separate this into a particular part and a homogeneous part:

$$\vec{y}(x) = \underbrace{\begin{pmatrix} x+2 \\ -2x \end{pmatrix}}_{\vec{y}_p(x)} + \underbrace{\begin{pmatrix} C_1e^x + C_2e^{3x} \\ -C_1e^x + C_2e^{3x} \end{pmatrix}}_{\vec{y}_h(x)}.$$

Therefore, $\vec{y} = \vec{y}_p + \vec{y}_h$ where

$$\vec{y}_p(x) = \begin{pmatrix} x+2 \\ -2x \end{pmatrix} \quad \text{and} \quad \vec{y}_h(x) = \begin{pmatrix} C_1e^x + C_2e^{3x} \\ -C_1e^x + C_2e^{3x} \end{pmatrix}.$$

25.4 Homogeneous systems

A fundamental system of solutions for a homogeneous linear system plays the same role as a fundamental set of solutions for a single linear differential equation. It consists of n linearly independent vector-valued solutions, where n is the size of the system. Since the space of all solutions has dimension n, such a family forms a basis of the whole solution space. Consequently, every solution of the system can be written uniquely as a linear combination of the solutions in the fundamental system. This is why finding a fundamental system is so important. We have the following analogous of Definition 13.6.

Definition 25.10. Consider a homogeneous system of linear ODEs

$$\vec{y}\,' = A\vec{y},$$

where $A \in \mathbb{R}^{n\times n}$. By a **fundamental system of solutions** of this system on an open interval I we mean any basis of the space of its solutions on I. If $\{\vec{y}_1, \dots, \vec{y}_n\}$ is a fundamental system of solutions, then we define its **fundamental matrix** on I by

$$Y(x) = \begin{pmatrix} \vec{y}_1(x) & \cdots & \vec{y}_n(x) \end{pmatrix},$$

which is an $n \times n$ matrix.

Example 25.11. As an example, consider the homogeneous system

$$\begin{cases} y_1' = 2y_1 + y_2, \\ y_2' = y_1 + 2y_2. \end{cases}$$

In matrix form, this system can be written as

$$\vec{y}\,' = A\vec{y},$$

where

$$\vec{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

From what we have done before, we know that a general solution of the associated nonhomogeneous system was given by

$$y_1 = x + 2 + C_1e^x + C_2e^{3x} \quad \text{and} \quad y_2 = -2x - C_1e^x + C_2e^{3x}.$$

Therefore, the general solution of the corresponding homogeneous system is

$$y_1 = C_1e^x + C_2e^{3x} \quad \text{and} \quad y_2 = -C_1e^x + C_2e^{3x}.$$

In vector form, this becomes

$$\vec{y}(x) = C_1 \begin{pmatrix} e^x \\ -e^x \end{pmatrix} + C_2 \begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix}.$$

Thus, two linearly independent solutions are

$$\vec{y}_1(x) = \begin{pmatrix} e^x \\ -e^x \end{pmatrix}, \qquad \vec{y}_2(x) = \begin{pmatrix} e^{3x} \\ e^{3x} \end{pmatrix}.$$

Hence, $\{\vec{y}_1, \vec{y}_2\}$ is a fundamental system of solutions, and the corresponding fundamental matrix is

$$Y(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix}. \tag{28}$$

We have the following fact.

Fact 25.12. Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in \mathbb{R}^{n\times n}$. If $Y(x)$ is its fundamental matrix on I, then a general solution of this system on I is

$$\vec{y}_h(x) = Y(x) \cdot \vec{c}$$

for $\vec{c} \in \mathbb{R}^n$.

For instance, in the example above we found that a fundamental matrix for the homogeneous system is given by (28). Therefore, by the previous fact, the general solution of the homogeneous system can be written as

$$\vec{y}_h(x) = Y(x) \cdot \vec{c},$$

where $\vec{c} \in \mathbb{R}^2$ is a constant vector. If we write

$$\vec{c} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

then

$$\vec{y}_h(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 e^x + C_2 e^{3x} \\ -C_1 e^x + C_2 e^{3x} \end{pmatrix}.$$

In order to verify that $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is a fundamental system of solutions what we need to do is to calculate a determinant. Indeed, we have the following result.

Theorem 25.13. Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$, where $A \in \mathbb{R}^{n\times n}$. Let $\vec{y}_1, \ldots, \vec{y}_n$ be solutions of this system on an open interval I. Then $\{\vec{y}_1, \ldots, \vec{y}_n\}$ is a fundamental system of solutions of this system on I if and only if $\det(Y(x)) \neq 0$ on I, which is true if and only if $\det(Y(x_0)) \neq 0$ for some $x_0 \in I$.

To check that the matrix (28) is indeed a fundamental matrix, we compute its determinant. We have

$$\det(Y(x)) = e^x \cdot e^{3x} - (-e^x) \cdot e^{3x}.$$

Hence,

$$\det(Y(x)) = e^{4x} + e^{4x} = 2e^{4x}.$$

Since $e^{4x} > 0$ for every $x \in \mathbb{R}$, it follows that

$$\det(Y(x)) = 2e^{4x} > 0$$

for all $x \in \mathbb{R}$. In particular, $\det(Y(x)) \neq 0$ for every x, and therefore the two column vectors of $Y(x)$ are linearly independent. This shows that $Y(x)$ is indeed a fundamental matrix of the system.