27 Practice #9: Wednesday, April 15th, 2026

27.1 Solving homogeneous systems of ODE

27.2 Problem 1

Let us solve the system

$$\begin{cases} y_1' = 2y_1 - y_2, \\ y_2' = 4y_1 - 3y_2, \end{cases}$$

with initial conditions $y_1(0) = 0$ and $y_2(0) = -3$. We first solve it by elimination. From the first equation, we isolate y_2:

$$y_2 = 2y_1 - y_1'.$$

We now differentiate the first equation. This gives

$$y_1'' = 2y_1' - y_2'.$$

Using the second equation, namely $y_2' = 4y_1 - 3y_2$, we obtain

$$y_1'' = 2y_1' - (4y_1 - 3y_2).$$

Substituting the expression $y_2 = 2y_1 - y_1'$ into this equation, we get

$$y_1'' = 2y_1' - 4y_1 + 3(2y_1 - y_1').$$

Expanding and simplifying, this becomes

$$y_1'' = -y_1' + 2y_1.$$

Hence, y_1 satisfies the second-order differential equation

$$y_1'' + y_1' - 2y_1 = 0.$$

We now solve this equation. The characteristic polynomial is

$$\lambda^2 + \lambda - 2 = 0.$$

Factoring, we obtain $(\lambda - 1)(\lambda + 2) = 0$, so the roots are $\lambda = 1$ and $\lambda = -2$. Therefore, the general solution for y_1 is

$$y_1(x) = C_1 e^x + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$. We now recover y_2 from the relation $y_2 = 2y_1 - y_1'$. Since

$$y_1'(x) = C_1 e^x - 2C_2 e^{-2x},$$

we get

$$y_2(x) = 2(C_1 e^x + C_2 e^{-2x}) - (C_1 e^x - 2C_2 e^{-2x}).$$

Thus,

$$y_2(x) = C_1 e^x + 4C_2 e^{-2x}$$

for every $x \in \mathbb{R}$. Let us now solve the same system by using the matrix approach. We write the system in matrix form as $\vec{y}\,' = A\vec{y}$, where the matrix of the system is given by

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}.$$

We now compute the eigenvalues of A. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{pmatrix}.$$

Hence,

$$\det(A - \lambda I) = (2-\lambda)(-3-\lambda) + 4.$$

Expanding, we get

$$\det(A - \lambda I) = \lambda^2 + \lambda - 2.$$

Therefore, the characteristic equation is $\lambda^2 + \lambda - 2 = 0$. Factoring, we obtain $(\lambda - 1)(\lambda + 2) = 0$, so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$. Let us now find an eigenvector associated with $\lambda_1 = 1$. We solve $(A - I)\vec{v} = 0$. Since

$$A - I = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix},$$

we obtain the equation $v_1 - v_2 = 0$. Thus, one possible eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, one solution of the system is

$$\vec{y}_1(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x.$$

Next, we find an eigenvector associated with $\lambda_2 = -2$. We solve $(A + 2I)\vec{v} = 0$. Since

$$A + 2I = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix},$$

we obtain the equation $4v_1 - v_2 = 0$. Thus, one possible eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Therefore, another solution of the system is

$$\vec{y}_2(x) = \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}.$$

Since the eigenvalues are distinct, these two solutions are linearly independent. Hence, the general solution of the system is

$$\vec{y}(x) = C_1\vec{y}_1(x) + C_2\vec{y}_2(x).$$

That is,

$$\vec{y}(x) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x + C_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}.$$

Therefore, once again we have

$$y_1(x) = C_1 e^x + C_2 e^{-2x} \quad \text{and} \quad y_2(x) = C_1 e^x + 4C_2 e^{-2x}.$$

We conclude by imposing the initial conditions. From $y_1(0) = 0$, we obtain $C_1 + C_2 = 0$. From $y_2(0) = -3$, we obtain $C_1 + 4C_2 = -3$. Since $C_1 = -C_2$, substituting into the second equation gives $-C_2 + 4C_2 = -3$. Hence, $3C_2 = -3$, so $C_2 = -1$ and therefore $C_1 = 1$. We conclude that

$$y_1(x) = e^x - e^{-2x} \quad \text{and} \quad y_2(x) = e^x - 4e^{-2x}.$$

Remark 27.1. Are the stationary solutions of this system stable? To determine the stationary solutions, we impose $y_1' = 0$ and $y_2' = 0$. This gives the system

$$\begin{cases} 2y_1 - y_2 = 0, \\ 4y_1 - 3y_2 = 0. \end{cases}$$

We have that $y_1 = 0$, and therefore also $y_2 = 0$. So the unique stationary solution is $\vec{y} = \vec{0}$. This stationary solution is unstable. Indeed, the general solution of the system is

$$\vec{y}(x) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x + C_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-2x}.$$

If $C_1 \neq 0$, then the term

$$C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^x$$

dominates as $x \to +\infty$. Since $e^x \to +\infty$ as $x \to +\infty$, it follows that $\vec{y}(x)$ does not remain close to $\vec{0}$, but instead moves away from the origin. Therefore, the stationary solution $\vec{y} = \vec{0}$ is unstable.

27.3 Problem 2

Example 27.2. As we have mentioned before, in many applications, especially in physics, engineering, control theory and dynamical systems, it is common to denote the independent variable by t rather than x, since t usually represents time. Likewise, one often writes the unknown vector as $\vec{x}(t)$ instead of $\vec{y}(x)$. With this notation, derivatives are frequently written using dots. For instance, $\dot{x}_1$ and $\dot{x}_2$ denote the derivatives of x_1 and x_2 with respect to t. Let us now solve the system

$$\begin{cases} \dot{x}_1 = x_1 + x_2, \\ \dot{x}_2 = -2x_1 + 3x_2. \end{cases}$$

We write the system in matrix form as

$$\dot{\vec{x}} = A\vec{x},$$

where

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}.$$

We now compute the eigenvalues of A. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{pmatrix}.$$

Hence, $\det(A - \lambda I) = (1-\lambda)(3-\lambda) + 2$. Expanding, we get $\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$. Therefore, the characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$. Its discriminant is

$$\Delta = (-4)^2 - 4 \cdot 1 \cdot 5 = 16 - 20 = -4.$$

Thus, the eigenvalues are

$$\lambda_1 = 2 + i \quad \text{and} \quad \lambda_2 = 2 - i.$$

Let us now find an eigenvector associated with $\lambda = 2 + i$. We solve $(A - (2+i)I)\vec{v} = 0$. Since

$$A - (2+i)I = \begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix},$$

the first row gives (notice that the second row is the first one multiplied by $(1-i)$)

$$(-1-i)v_1 + v_2 = 0.$$

Hence,

$$v_2 = (1+i)v_1.$$

Taking $v_1 = 1$, we may choose

$$\vec{v} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

Therefore, a complex-valued solution is

$$\vec{x}_c(t) = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2+i)t}.$$

Using Euler's formula, we can write

$$e^{(2+i)t} = e^{2t}(\cos t + i \sin t).$$

Thus,

$$\vec{x}_c(t) = e^{2t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} (\cos t + i \sin t).$$

We now extract the real and imaginary parts. First component is $e^{2t}(\cos t + i \sin t)$. So its real part is $e^{2t} \cos t$ and its imaginary part is $e^{2t} \sin t$. For the second component, we compute

$$(1+i)(\cos t + i \sin t) = \cos t + i \sin t + i \cos t - \sin t.$$

Hence,

$$(1+i)(\cos t + i \sin t) = (\cos t - \sin t) + i(\sin t + \cos t).$$

Therefore, the real and imaginary parts give the two real solutions

$$\mathrm{Re}(\vec{x}_c(t)) = e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} \quad \text{and} \quad \mathrm{Im}(\vec{x}_c(t)) = e^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}.$$

These two solutions are linearly independent. Hence, the general solution of the system is

$$\vec{x}(t) = C_1 \vec{x}^{(1)}(t) + C_2 \vec{x}^{(2)}(t).$$

That is,

$$\vec{x}(t) = C_1 e^{2t} \begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin t \\ \sin t + \cos t \end{pmatrix}.$$

Therefore,

$$x_1(t) = C_1 \cos(t) e^{2t} + C_2 \sin(t) e^{2t},$$

and

$$x_2(t) = C_1(\cos(t) - \sin(t))e^{2t} + C_2(\sin(t) + \cos(t))e^{2t}$$

for every $x \in \mathbb{R}$.

27.4 Problem 3

Example 27.3. Let us solve the initial value problem

$$\begin{cases} y_1' = 2y_1 - y_2, \\ y_2' = -2y_1 + y_2, \end{cases}$$

with initial conditions $y_1(1) = 1 + 2e^3$ and $y_2(1) = 2 + e^3$. We solve it by using the matrix approach. We first write the system in matrix form $\vec{y}' = A\vec{y}$ where the matrix of the system is given by

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}.$$

We start by computing the eigenvalues of A. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2 - \lambda & -1 \\ -2 & 1 - \lambda \end{pmatrix}.$$

Hence,

$$\det(A - \lambda I) = (2 - \lambda)(1 - \lambda) - 2.$$

Expanding, we get $\det(A - \lambda I) = \lambda^2 - 3\lambda$. Therefore, the characteristic equation is $\lambda^2 - 3\lambda = 0$. Factoring, we obtain $\lambda(\lambda - 3) = 0$, so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. Let us now find an eigenvector associated with $\lambda_1 = 0$. We solve $A\vec{v} = 0$. We obtain the equation

$$2v_1 - v_2 = 0.$$

Thus, one possible eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Therefore, one solution of the system is

$$\vec{y}_1(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Next, we find an eigenvector associated with $\lambda_2 = 3$. We solve $(A - 3I)\vec{v} = 0$. Since

$$A - 3I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix},$$

we obtain the equation $v_1 + v_2 = 0$. Thus, one possible eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, another solution of the system is

$$\vec{y}_2(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3x}.$$

Since the eigenvalues are distinct, these two solutions are linearly independent. Hence, the general solution of the system is

$$\vec{y}(x) = C_1\vec{y}_1(x) + C_2\vec{y}_2(x).$$

That is,

$$\vec{y}(x) = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3x}.$$

Therefore,

$$y_1(x) = C_1 + C_2e^{3x} \quad \text{and} \quad y_2(x) = 2C_1 - C_2e^{3x}$$

for every $x \in \mathbb{R}$. We now impose the initial conditions at $x = 1$. This gives $C_1 + C_2e^3 = 1 + 2e^3$ and $2C_1 - C_2e^3 = 2 + e^3$. Solving this system, we get that $C_1 = 1 + e^3$ and $C_2 = 1$. We then conclude that

$$\vec{y}(x) = (1 + e^3) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3x}.$$

Thus,

$$y_1(x) = 1 + e^3 + e^{3x} \quad \text{and} \quad y_2(x) = 2 + 2e^3 - e^{3x}$$

for every $x \in \mathbb{R}$.

27.5 Problem 4

Let us convert the differential equation $y''' + 3y'' - 2y' - y = 0$ into a first-order system. We introduce new unknown functions by setting $y_1 = y$, $y_2 = y_1' = y'$, $y_3 = y_2' = y''$. We now compute their derivatives. Since $y_1 = y$, we have $y_1' = y' = y_2$. Since $y_2 = y'$, we have $y_2' = y'' = y_3$. Finally, since $y_3 = y''$, we have $y_3' = y'''$. From the original differential equation,

$$y''' + 3y'' - 2y' - y = 0,$$

we obtain

$$y''' = -3y'' + 2y' + y.$$

Using the definitions of y_1, y_2, and y_3, this becomes

$$y_3' = -3y_3 + 2y_2 + y_1.$$

Therefore, the equation is equivalent to the system

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = y_1 + 2y_2 - 3y_3. \end{cases}$$