26 Lecture #18: Wednesday, April 15th, 2026

26.1 How to find a fundamental matrix

From what we have seen before, we need to find a way to find the fundamental matrix. For this, we need a tool from Linear Algebra.

Definition 26.1. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. A number λ is called an **eigenvalue** of A if there exists a non-zero vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \lambda\vec{x}$. Vectors $\vec{x}$ with this property are then called **eigenvectors** of A associated with (or corresponding to) the eigenvalue λ.

The key statement is the following one.

Theorem 26.2. Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n\times n}$. If λ_0 is an eigenvalue of A with associated eigenvector $\vec{v}$, then

$$\vec{y} = \vec{v}\, e^{\lambda_0 x}$$

is a solution of the given system on $\mathbb{R}$. If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of the matrix A, then the corresponding solutions form a linearly independent set.

Example 26.3. Let us now solve the homogeneous system

$$\begin{cases} y_1' = 2y_1 + y_2, \\ y_2' = y_1 + 2y_2. \end{cases}$$

by using eigenvalues and eigenvectors. We first write the system in matrix form:

$$\vec{y}\,' = A\vec{y},$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We now compute the eigenvalues of A. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}.$$

Hence,

$$\det(A - \lambda I) = (2-\lambda)^2 - 1.$$

Expanding, we get

$$(2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3.$$

Therefore, the characteristic equation is

$$\lambda^2 - 4\lambda + 3 = 0.$$

Factoring, we obtain

$$(\lambda - 1)(\lambda - 3) = 0,$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Let us now find an eigenvector associated with $\lambda_1 = 1$. We solve $(A - I)\vec{v} = 0$. Since

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

we obtain the equation $v_1 + v_2 = 0$. Thus, one possible eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, one solution of the system is

$$\vec{y}_1(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^x.$$

Next, we find an eigenvector associated with $\lambda_2 = 3$. We solve $(A - 3I)\vec{v} = 0$. Since

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

we obtain the equation $-v_1 + v_2 = 0$. Thus, one possible eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, another solution of the system is

$$\vec{y}_2(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x}.$$

Since the eigenvalues are distinct, these two solutions are linearly independent. Hence, the general solution of the system is $\vec{y}(x) = C_1\vec{y}_1(x) + C_2\vec{y}_2(x)$. That is,

$$\vec{y}(x) = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^x + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3x}.$$

Therefore, $y_1(x) = C_1e^x + C_2e^{3x}$ and $y_2(x) = -C_1e^x + C_2e^{3x}$.

Remark 26.4. In the previous example, the stationary solutions are obtained by imposing $y_1' = 0$ and $y_2' = 0$. This leads to the system

$$\begin{cases} 2y_1 + y_2 = 0, \\ y_1 + 2y_2 = 0. \end{cases}$$

whose only solution is $y_1 = y_2 = 0$. Hence, the origin is the unique stationary solution of the system. A natural question is then: are they stable? In this case, the answer is no. Indeed, the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

are $\lambda_1 = 1$ and $\lambda_2 = 3$, both of which are positive. Therefore, the general solution contains the terms e^x and e^{3x}, which grow exponentially as $x \to +\infty$. This means that solutions starting close to the stationary point $(0, 0)$ move away from it rather than approaching it. Consequently, the stationary solution $(0, 0)$ is unstable.

The next example shows that not always we get such a smooth situation as in the previous example.

Example 26.5. As we know, the equation $y'' + 4y = 0$ can be solved by using the characteristic polynomial. Instead. let us transform it into a system of first-order equations and apply the method using the fundamental matrix. We introduce the new variables $y_1 = y$ and $y_2 = y_1'$. Since $y_2' = y''$, the original equation gives $y_2' = -4y = -4y_1$. Therefore, the equation can be written as the system

$$\begin{cases} y_1' = y_2, \\ y_2' = -4y_1. \end{cases}$$

In matrix form, this becomes $\vec{y}\,' = A\vec{y}$, where the matrix of the system is given by

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}.$$

We now find the eigenvalues of the matrix A. Its characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -4 & -\lambda \end{pmatrix} = \lambda^2 + 4.$$

Therefore, the characteristic equation is $\lambda^2 + 4 = 0$, and hence the eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Let us now find an eigenvector associated with $\lambda_1 = 2i$. We solve $(A - 2iI)\vec{v} = 0$. Since

$$A - 2iI = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}$$

the equation $-2iv_1 + v_2 = 0$ gives $v_2 = 2iv_1$. Taking $v_1 = 1$, we obtain the eigenvector

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

In the same way, for $\lambda_2 = -2i$, we solve $(A + 2iI)\vec{v} = 0$, and obtain $v_2 = -2iv_1$. Taking again $v_1 = 1$, we find the eigenvector

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -2i \end{pmatrix}.$$

Thus, the corresponding complex-valued solutions are

$$\vec{y}_1(x) = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2ix} \quad \text{and} \quad \vec{y}_2(x) = \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2ix}.$$

These solutions can then be combined to obtain the usual real-valued solutions of the equation. We do not particularly like the solution in this form, since it is written in terms of complex exponentials, whereas in most applications we are interested in real-valued solutions. For this reason, we now rewrite the complex solution in a more useful way. Consider the solution associated with the eigenvalue $\lambda = 2i$ and eigenvector

$$\vec{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}.$$

Then

$$\vec{v}\, e^{2ix} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2ix}.$$

Using Euler's formula,

$$e^{2ix} = \cos(2x) + i\sin(2x),$$

we can write

$$\vec{v}\, e^{2ix} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \big(\cos(2x) + i\sin(2x)\big).$$

Developing this expression componentwise, we obtain

$$\vec{v}\, e^{2ix} = \begin{pmatrix} \cos(2x) + i\sin(2x) \\ 2i\cos(2x) + 2i^2\sin(2x) \end{pmatrix}.$$

Since $i^2 = -1$, this becomes

$$\vec{v}\, e^{2ix} = \begin{pmatrix} \cos(2x) + i\sin(2x) \\ -2\sin(2x) + 2i\cos(2x) \end{pmatrix}.$$

Therefore, separating real and imaginary parts, we get

$$\vec{v}\, e^{2ix} = \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + i \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}.$$

This is much better for our purposes, because it allows us to extract two real-valued solutions of the system, namely

$$\begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}.$$

It is not necessary to repeat the same computation for the second eigenvector. Indeed, the second eigenvalue $\lambda = -2i$ and its corresponding eigenvector are just the complex conjugates of the first ones. Therefore, the associated solution is the complex conjugate of the solution obtained from $\lambda = 2i$. As a consequence, it does not provide any new real-valued solutions: it leads to the same real and imaginary parts as before. For this reason, it is enough to work with only one complex eigenvalue-eigenvector pair in order to obtain the real-valued solutions of the system.

Therefore, from

$$\vec{v}\, e^{2ix} = \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + i \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix},$$

we obtain two real-valued solutions

$$\vec{y}_1(x) = \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} \quad \text{and} \quad \vec{y}_2(x) = \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}.$$

Hence, the general real solution of the system is

$$\vec{y}(x) = C_1\vec{y}_1(x) + C_2\vec{y}_2(x),$$

that is,

$$\vec{y}(x) = C_1 \begin{pmatrix} \cos(2x) \\ -2\sin(2x) \end{pmatrix} + C_2 \begin{pmatrix} \sin(2x) \\ 2\cos(2x) \end{pmatrix}. \tag{29}$$

Equivalently,

$$y_1(x) = C_1\cos(2x) + C_2\sin(2x),$$

and

$$y_2(x) = -2C_1\sin(2x) + 2C_2\cos(2x).$$

As $y = y_1$, the first equation is the general solution of the original equation $y'' + 4y = 0$.

The strategy we have used in the previous example always works as we can see in the following result.

Fact 26.6. Consider a homogeneous system of linear ODEs $\vec{y}\,' = A\vec{y}$ with matrix $A \in \mathbb{R}^{n\times n}$. Let λ_0 be an eigenvalue of A with associated eigenvector $\vec{v}$. If λ_0 is a complex number, that is, $\mathrm{Im}(\lambda_0) \neq 0$, then $\mathrm{Re}(\vec{v}e^{\lambda_0 x})$ and $\mathrm{Im}(\vec{v}e^{\lambda_0 x})$ are linearly independent solutions of the given system on $\mathbb{R}$.

Remark 26.7. A natural question is whether the stationary solution $\vec{y} = \vec{0}$ is stable. By taking a quick look at the solutions in (29), we see that this is not the case. Thus, once again, we obtain an unstable stationary solution. We will learn how to do it in the next example.

It remains to consider the case in which an eigenvalue has multiplicity greater than one. To handle this situation, we use the following result, which gives the required form of the solutions, although at first sight it may not look especially friendly.

Fact 26.8. Consider a homogeneous system of linear ODEs

$$\vec{y}\,' = A\vec{y}$$

with matrix $A \in \mathbb{R}^{n\times n}$. Let λ_0 be an eigenvalue of A of multiplicity m with associated eigenvector $\vec{v}$. Consider vectors defined as follows:

$$\begin{gathered}
\vec{v}_1 = \vec{v},\\
\vec{v}_2 \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_1,\\
\vec{v}_3 \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_2,\\
\vdots\\
\vec{v}_m \text{ is a solution of } (A - \lambda_0 E_n)\vec{x} = \vec{v}_{m-1}.
\end{gathered}$$

Then the following functions are solutions of the given system on $\mathbb{R}$ and form a linearly independent set:

$$\begin{gathered}
\vec{y} = \vec{v}_1 e^{\lambda_0 x},\\
\vec{y} = \left[\int (\vec{v}_1)\,dx + \vec{v}_2\right] e^{\lambda_0 x} = (\vec{v}_1 x + \vec{v}_2)e^{\lambda_0 x},\\
\vec{y} = \left[\int (\vec{v}_1 x + \vec{v}_2)\,dx + \vec{v}_3\right] e^{\lambda_0 x} = \left(\frac{1}{2}\vec{v}_1 x^2 + \vec{v}_2 x + \vec{v}_3\right) e^{\lambda_0 x},\\
\vdots\\
\vec{y} = \left(\frac{1}{(m-1)!}\vec{v}_1 x^{m-1} + \frac{1}{(m-2)!}\vec{v}_2 x^{m-2} + \cdots + \vec{v}_{m-1}x + \vec{v}_m\right) e^{\lambda_0 x}.
\end{gathered}$$

Example 26.9. Let us solve the system

$$\begin{cases} y_1' = -y_1 + y_2,\\ y_2' = -y_1 - 3y_2, \end{cases}$$

with initial conditions $y_1(0) = 13$ and $y_2(0) = 23$. We first write the system in matrix form $\vec{y}\,' = A\vec{y}$, where

$$A = \begin{pmatrix} -1 & 1\\ -1 & -3 \end{pmatrix}.$$

We now compute the eigenvalues of A. The characteristic polynomial is

$$\det(A-\lambda I)=\det\begin{pmatrix}-1-\lambda & 1\\ -1 & -3-\lambda\end{pmatrix}.$$

Hence, $\det(A-\lambda I)=(-1-\lambda)(-3-\lambda)+1$. Expanding, we obtain

$$(-1-\lambda)(-3-\lambda)+1=(\lambda+1)(\lambda+3)+1=\lambda^2+4\lambda+4.$$

Therefore, the characteristic equation is $\lambda^2+4\lambda+4=0$, that is, $(\lambda+2)^2=0$. So the matrix has a repeated eigenvalue $\lambda_0=-2$ of multiplicity 2. We now find an eigenvector associated with $\lambda_0=-2$. We solve $(A+2I)\vec{v}_1=0$. Since

$$A+2I=\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix},$$

we obtain the equation $v_{1,1}+v_{1,2}=0$. Thus, one possible eigenvector is

$$\vec{v}_1=\begin{pmatrix}1\\ -1\end{pmatrix}.$$

Therefore, one solution of the system is

$$\vec{y}_1(x)=\vec{v}_1e^{-2x}=\begin{pmatrix}1\\ -1\end{pmatrix}e^{-2x}.$$

Since the eigenvalue has multiplicity 2, we now look for a generalized eigenvector $\vec{v}_2$ satisfying

$$(A+2I)\vec{v}_2=\vec{v}_1$$

Writing

$$\vec{v}_2=\begin{pmatrix}a\\ b\end{pmatrix},$$

this becomes

$$\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1\\ -1\end{pmatrix}.$$

So we get $a+b=1$. We may choose, for instance,

$$\vec{v}_2=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

By the result for repeated eigenvalues, a second solution is then $\vec{y}_2(x)=\Big(\vec{v}_1x+\vec{v}_2\Big)e^{-2x}$. Substituting the vectors found above, we obtain

$$\vec{y}_2(x)=\left(\begin{pmatrix}1\\ -1\end{pmatrix}x+\begin{pmatrix}1\\ 0\end{pmatrix}\right)e^{-2x}=\begin{pmatrix}x+1\\ -x\end{pmatrix}e^{-2x}.$$

Hence, the general solution of the system is $\vec{y}(x)=C_1\vec{y}_1(x)+C_2\vec{y}_2(x)$. That is,

$$\vec{y}(x)=C_1\begin{pmatrix}1\\ -1\end{pmatrix}e^{-2x}+C_2\begin{pmatrix}x+1\\ -x\end{pmatrix}e^{-2x}.$$

Therefore, $y_1(x)=\Big(C_1+C_2(x+1)\Big)e^{-2x}$ and $y_2(x)=\Big(-C_1-C_2x\Big)e^{-2x}$ for every $x\in\mathbb{R}$. We now impose the initial conditions. Since $y_1(0)=13$, we get $C_1+C_2=13$. Since $y_2(0)=23$, we obtain $-C_1=23$. Thus, $C_1=-23$. Substituting into the first equation, we find $-23+C_2=13$, so $C_2=36$. Hence, the solution of the initial value problem is $y_1(x)=\Big(36x+13\Big)e^{-2x}$ and $y_2(x)=\Big(23-36x\Big)e^{-2x}$.

26.2 Non-homogeneous systems of equations

We now turn to non-homogeneous systems of equations. After studying homogeneous systems, this is the next natural step, since in many applications the evolution of the unknown vector is influenced not only by the interaction between its components, but also by an external forcing term. In matrix form, such systems are written as

$$\vec{y}\,' = A\vec{y} + \vec{b}(x)$$

where $\vec{b}(x) \neq \vec{0}$. As in the scalar case, the general philosophy is to first understand the associated homogeneous system (which we did) and then look for a particular solution of the full one. In this way, the theory of non-homogeneous systems mirrors very closely what we already know for linear differential equations of first order. Before treating systems, let us briefly recall how one solves a linear equation of the form $y' + a(x)y = b(x)$. The first step is to solve the associated homogeneous equation $y' + a(x)y = 0$, whose general solution is $y_h(x) = Ce^{-A(x)}$, where $A'(x) = a(x)$. Then, instead of taking the constant C, one allows it to vary with x and writes $y_p(x) = C(x)e^{-A(x)}$. Substituting this expression into the equation, one obtains an equation for $C'(x)$, which can then be integrated. As we have seen, this method is called variation of constants (or variation of parameters), and it will serve as the model for the corresponding method for systems.

26.3 The variant method

Let us now explain how the method of variation of constants works for a non-homogeneous system of the form $\vec{y}\,' = A\vec{y} + \vec{b}(x)$. The idea is exactly the same as in the scalar case with some suitable modifications. We first solve the associated homogeneous system and then allow the constants to vary with x. More precisely, the method goes by following the next ten steps:

1. First, solve the associated homogeneous system $\vec{y}\,' = A\vec{y}$. If $Y(x)$ is a fundamental matrix of this homogeneous system, then its general solution is given by

$$\vec{y}_h(x) = Y(x) \cdot \vec{c}$$

where $\vec{c} \in \mathbb{R}^n$ is a constant vector.

2. In the non-homogeneous case, instead of taking $\vec{c}$ constant, we let it depend on x. That is, we look for a solution of the form

$$\vec{y}_p(x) = Y(x) \cdot \vec{c}(x),$$

where $\vec{c}(x)$ is now an unknown vector-valued function.

3. We then differentiate this expression. Using the product rule, we get

$$\vec{y}_p'(x) = Y'(x) \cdot \vec{c}(x) + Y(x) \cdot \vec{c}\,'(x).$$

4. Since $Y(x)$ is a fundamental matrix of the homogeneous system, it satisfies $Y'(x) = AY(x)$. Therefore,

$$\vec{y}_p'(x) = A \cdot Y(x) \cdot \vec{c}(x) + Y(x) \cdot \vec{c}\,'(x).$$

5. On the other hand, because $\vec{y}_p$ must satisfy the non-homogeneous system, we also have

$$\vec{y}_p'(x) = A \cdot \vec{y}_p(x) + \vec{b}(x) = A \cdot Y(x) \cdot \vec{c}(x) + \vec{b}(x).$$

6. Comparing the two expressions for $\vec{y}_p'(x)$, the terms $A \cdot Y(x) \cdot \vec{c}(x)$ cancel and we are left with

$$Y(x) \cdot \vec{c}\,'(x) = \vec{b}(x).$$

7. Since $Y(x)$ is a fundamental matrix, it is invertible. Hence we can multiply by $Y(x)^{-1}$ and obtain

$$\vec{c}\,'(x) = Y(x)^{-1} \cdot \vec{b}(x).$$

8. Integrating (and this means integrating every row of the matrix), we find

$$\vec{c}(x) = \int Y(x)^{-1} \cdot \vec{b}(x)\, dx.$$

9. Substituting this back into the expression for $\vec{y}_p$, we obtain a particular solution:

$$\vec{y}_p(x) = Y(x) \cdot \int Y(x)^{-1}\vec{b}(x)\, dx.$$

10. Finally, the general solution of the non-homogeneous system is obtained by adding the homogeneous part and a particular solution:

$$\vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x) = Y(x) \cdot \vec{c} + Y(x) \cdot \int Y(x)^{-1} \cdot \vec{b}(x)\, dx.$$

So, in summary, variation of constants for systems follows exactly the same principle as in the scalar case: we replace the constant vector in the homogeneous solution by a variable vector and then determine it by substitution.

Example 26.10. By applying this method, let us solve the system

$$\begin{cases} y_1' = 2y_1 + y_2 - 3, \\ y_2' = y_1 + 2y_2 + 3x - 4. \end{cases}$$

We already know that the associated homogeneous system has fundamental matrix

$$Y(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix},$$

so that its general solution is

$$\vec{y}_h(x) = Y(x) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

for every $x \in \mathbb{R}$. We now solve the nonhomogeneous system by variation of constants. We look for a particular solution in the form

$$\vec{y}_p(x) = Y(x)\vec{c}(x),$$

where

$$\vec{c}(x) = \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}.$$

The system can be written as $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b}(x) = \begin{pmatrix} -3 \\ 3x - 4 \end{pmatrix}.$$

By the variation method, $\vec{c}(x)$ must satisfy $Y(x)\vec{c}\,'(x) = \vec{b}(x)$. Therefore,

$$\vec{c}\,'(x) = Y(x)^{-1}\vec{b}(x).$$

Let us compute $Y(x)^{-1}$. First,

$$\det Y(x) = \det \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix} = 2e^{4x}.$$

Hence,

$$Y(x)^{-1} = \frac{1}{2e^{4x}} \begin{pmatrix} e^{3x} & -e^{3x} \\ e^x & e^x \end{pmatrix}.$$

Thus,

$$\vec{c}\,'(x) = \frac{1}{2e^{4x}} \begin{pmatrix} e^{3x} & -e^{3x} \\ e^x & e^x \end{pmatrix} \begin{pmatrix} -3 \\ 3x-4 \end{pmatrix}.$$

Carrying out the multiplication, we get

$$\vec{c}\,'(x) = \begin{pmatrix} \dfrac{1-3x}{2}e^{-x} \\ \dfrac{3x-7}{2}e^{-3x} \end{pmatrix}.$$

We now integrate component by component. For the first one,

$$c_1(x) = \int \frac{1-3x}{2}e^{-x}\,dx = \frac{3x+2}{2}e^{-x}.$$

For the second one,

$$c_2(x) = \int \frac{3x-7}{2}e^{-3x}\,dx = \frac{2-x}{2}e^{-3x}.$$

Therefore, we may take

$$\vec{c}(x) = \begin{pmatrix} \dfrac{3x+2}{2}e^{-x} \\ \dfrac{2-x}{2}e^{-3x} \end{pmatrix}.$$

We now compute the particular solution as follows

$$\vec{y}_p(x) = Y(x)\vec{c}(x).$$

Hence,

$$\vec{y}_p(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix} \begin{pmatrix} \dfrac{3x+2}{2}e^{-x} \\ \dfrac{2-x}{2}e^{-3x} \end{pmatrix}.$$

Multiplying, we obtain

$$y_{1,p}(x) = \frac{3x+2}{2} + \frac{2-x}{2} = x+2,$$

and

$$y_{2,p}(x) = -\frac{3x+2}{2} + \frac{2-x}{2} = -2x.$$

Thus,

$$\vec{y}_p(x) = \begin{pmatrix} x+2 \\ -2x \end{pmatrix}.$$

Finally, the general solution of the nonhomogeneous system is

$$\vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x).$$

That is,

$$\vec{y}(x) = Y(x)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} x+2 \\ -2x \end{pmatrix}.$$

Equivalently,

$$\vec{y}(x) = \begin{pmatrix} e^x & e^{3x} \\ -e^x & e^{3x} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} x+2 \\ -2x \end{pmatrix}.$$

Therefore, the solutions are given by

$$y_1(x) = C_1 e^x + C_2 e^{3x} + x + 2 \quad \text{and} \quad y_2(x) = -C_1 e^x + C_2 e^{3x} - 2x$$

for every $x \in \mathbb{R}$.

26.4 The row variation

Formally, the variation of parameters method for systems can be summarized in the following algorithm. Observe that, in the previous example (see Example 26.10), we applied part (b) below, namely the vector variation approach, whereas in the next example (see Example 26.12) we shall use part (a), that is, the row variation approach.

Algorithm 26.11 (variation of parameters method)**.** Consider a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$.

1. Find a general solution $\vec{y}_h$ of the associated homogeneous system $\vec{y}\,' = A\vec{y}$:

$$y_{1h}(x) = c_1 u_1(x) + c_2 v_1(x) + \cdots,$$

$$y_{2h} = \ldots,$$

$$y_{nh}(x) = c_1 u_n(x) + c_2 v_n(x) + \cdots.$$

2. **a) Row variation:** We seek solution of the form

$$y_1(x) = c_1(x)u_1(x) + c_2(x)v_1(x) + \cdots,$$

$$\vdots$$

$$y_n(x) = c_1(x)u_n(x) + c_2(x)v_n(x) + \cdots.$$

Unknown functions $c_i(x)$ are found by solving the system of equations

$$c_1'(x)u_1(x) + c_2'(x)v_1(x) + \cdots = b_1(x),$$

$$\vdots$$

$$c_1'(x)u_n(x) + c_2'(x)v_n(x) + \cdots = b_n(x).$$

From here determine (using e.g. elimination or Cramer rule)

$$c_1'(x), \dots, c_n'(x),$$

integrating them one gets

$$c_1(x), \dots, c_n(x).$$

Substitute these into modified $y_1, \dots, y_n$ to get

$$y_{1p}, \dots, y_{np}.$$

The general solution is

$$\vec{y}_i = \vec{y}_p + \vec{y}_h.$$

3. **b) Vector variation:** We write the homogeneous solutions as

$$\vec{y}_h = Y(x) \cdot \vec{c}.$$

We seek solution of the form

$$\vec{y} = Y(x) \cdot \vec{c}(x).$$

Solve the equation

$$Y(x) \cdot \vec{c}\,'(x) = \vec{b}(x)$$

for

$$\vec{c}\,'(x) = Y(x)^{-1}\vec{b}(x).$$

Integrating by rows get $\vec{c}(x)$ and substitute into

$$\vec{y}(x) = Y(x) \cdot \vec{c}(x).$$

This yields $\vec{y}_p$, the general solution is then

$$\vec{y} = \vec{y}_p + \vec{y}_h.$$

In the next example, we solve the same system as in the previous example by using the row variation method, and we shall see that this approach leads to the same solution. It is then up to the reader to decide which of the two methods is more convenient in practice.

Example 26.12. Let us solve the system

$$\begin{cases} y_1' = 2y_1 + y_2 - 3, \\ y_2' = y_1 + 2y_2 + 3x - 4. \end{cases}$$

by using the row variation method. We already know that the associated homogeneous system has general solution

$$y_{1h}(x) = C_1e^x + C_2e^{3x} \quad \text{and} \quad y_{2h}(x) = -C_1e^x + C_2e^{3x}.$$

We now replace the constants C_1 and C_2 by functions $C_1(x)$ and $C_2(x)$. Thus, we seek a solution of the form

$$y_1(x) = C_1(x)e^x + C_2(x)e^{3x},$$

$$y_2(x) = -C_1(x)e^x + C_2(x)e^{3x}.$$

By the row variation method, the unknown functions $C_1(x)$ and $C_2(x)$ are determined from the system

$$C_1'(x)e^x + C_2'(x)e^{3x} = -3,$$

$$-C_1'(x)e^x + C_2'(x)e^{3x} = 3x - 4.$$

At this point, one solves this 2×2 system for $C_1'(x)$ and $C_2'(x)$. After that, one integrates to obtain $C_1(x)$ and $C_2(x)$. Substituting these functions into

$$y_1(x) = C_1(x)e^x + C_2(x)e^{3x} \quad \text{and} \quad y_2(x) = -C_1(x)e^x + C_2(x)e^{3x},$$

one gets a particular solution. In this case, the computation leads once again as in the previous example to

$$\vec{y}_p(x) = \begin{pmatrix} x+2 \\ -2x \end{pmatrix}.$$

Therefore, the general solution of the system is

$$\vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x).$$

That is,

$$\vec{y}(x) = \begin{pmatrix} C_1e^x + C_2e^{3x} + x + 2 \\ -C_1e^x + C_2e^{3x} - 2x \end{pmatrix}.$$

Hence,

$$y_1(x) = C_1e^x + C_2e^{3x} + x + 2 \quad \text{and} \quad y_2(x) = -C_1e^x + C_2e^{3x} - 2x.$$