33 Practice #11: Wednesday, April 29th, 2026

33.1 Solving systems of linear equations by elimination

33.2 Problem 1

Let us consider the system of linear equations

$$\begin{cases} 2x+4y-2z=4, \\ x+y-2z=3, \\ -3x+y+8z=-11. \end{cases}$$

We first write the system in matrix form as follows

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & -2 & 3 \\ -3 & 1 & 8 & -11 \end{array}\right).$$

Now, using the notation (#1), (#2) and (#3) for the three rows, we perform the following operations indicated below

$$\begin{aligned} (\#3)+(\#1)+(\#2) &\longrightarrow (\#3), \\ 2\cdot(\#2)-(\#1) &\longrightarrow (\#2), \\ \tfrac{1}{2}\cdot(\#1) &\longrightarrow (\#1). \end{aligned}$$

This gives

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & -2 & 3 \\ -3 & 1 & 8 & -11 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -2 & -2 & 2 \\ 0 & 6 & 4 & -4 \end{array}\right).$$

So the new system is

$$\begin{cases} x+2y-z=2, \\ -2y-2z=2, \\ 6y+4z=-4. \end{cases}$$

We can continue by using the exact same strategy until we get a triangular form for the system. However, this is not the kind of elimination we allow in Numerical Analysis. The first two operations are not admissible, because they involve more than one row at once. In Gaussian elimination, one is only allowed to replace a row by that row plus or minus a multiple of a single pivot row. In particular, the second operation multiplies row (#2) by 2 and then subtracts row (#1), which is not part of the standard elimination procedure. The only admissible operation here is the third one, namely multiplying row (#1) by $\frac{1}{2}$. We **do not want** this kind of manipulation in the test.

Now we will procedure with the solution of the system by applying the Gauss-Jordan approach. Once again we consider the matrix

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & -2 & 3 \\ -3 & 1 & 8 & -11 \end{array}\right).$$

In the first stage, we perform the operations

$$\begin{aligned} &(\#1) \leftrightarrow (\#2), \\ (\#2) - 2\cdot(\#1) &\rightarrow (\#2), \\ (\#3) + 3\cdot(\#1) &\rightarrow (\#3). \end{aligned}$$

Thus,

$$\left(\begin{array}{ccc|c} 2 & 4 & -2 & 4 \\ 1 & 1 & -2 & 3 \\ -3 & 1 & 8 & -11 \end{array}\right) \xrightarrow{(\#1)\leftrightarrow(\#2)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & 4 & -2 & 4 \\ -3 & 1 & 8 & -11 \end{array}\right)$$

$$\xrightarrow{(\#2)-2\cdot(\#1)\rightarrow(\#2)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ -3 & 1 & 8 & -11 \end{array}\right) \xrightarrow{(\#3)+3\cdot(\#1)\rightarrow(\#3)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 2 & -2 \end{array}\right).$$

In the second stage, we perform the operations

$$\begin{aligned} \tfrac{1}{2}\cdot(\#2) &\rightarrow (\#2), \\ (\#1) - (\#2) &\rightarrow (\#1), \\ (\#3) - 4\cdot(\#2) &\rightarrow (\#3). \end{aligned}$$

Hence,

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 2 & -2 \end{array}\right) \xrightarrow{\frac{1}{2}\cdot(\#2)\rightarrow(\#2)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 2 & -2 \end{array}\right)$$

$$\xrightarrow{(\#1)-(\#2)\rightarrow(\#1)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 4 & 2 & -2 \end{array}\right) \xrightarrow{(\#3)-4\cdot(\#2)\rightarrow(\#3)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \end{array}\right).$$

Finally, we perform the operations

$$\begin{aligned} -\tfrac{1}{2}\cdot(\#3) &\rightarrow (\#3), \\ (\#2) - (\#3) &\rightarrow (\#2), \\ (\#1) + 3\cdot(\#3) &\rightarrow (\#1). \end{aligned}$$

Therefore,

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \end{array}\right) \xrightarrow{-\frac{1}{2}\cdot(\#3)\rightarrow(\#3)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{(\#2)-(\#3)\rightarrow(\#2)} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{(\#1)+3\cdot(\#3)\rightarrow(\#1)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

Thus, the solution is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Let us analyze now how many flops were needed to obtain this solution. We shall count them manually. The first stage consists of the operations

$$\begin{array}{llll} (\#1) \leftrightarrow (\#2), & (0 \text{ flops}) & & \\ (\#2) - 2 \cdot (\#1) & \to & (\#2), & (1 + 3 \cdot 2 = 7 \text{ flops}), \\ (\#3) + 3 \cdot (\#1) & \to & (\#3), & (1 + 3 \cdot 2 = 7 \text{ flops}). \end{array}$$

Altogether, this first stage requires 14 flops. Now let us consider the second stage:

$$\begin{array}{rlll} \frac{1}{2} \cdot (\#2) & \to & (\#2), & (3 \text{ flops}), \\ (\#1) - 1 \cdot (\#2) & \to & (\#1), & (1 + 1 \cdot 2 = 3 \text{ flops}), \\ (\#3) - 4 \cdot (\#2) & \to & (\#3), & (1 + 2 \cdot 2 = 5 \text{ flops}). \end{array}$$

Hence, the second stage takes 11 flops in total. Finally, in the last stage we have

$$\begin{array}{rlll} -\frac{1}{2} \cdot (\#3) & \to & (\#3), & (2 \text{ flops}), \\ (\#2) - 1 \cdot (\#3) & \to & (\#2), & (1 + 1 \cdot 1 = 2 \text{ flops}), \\ (\#1) + 3 \cdot (\#3) & \to & (\#1), & (1 + 1 \cdot 2 = 3 \text{ flops}). \end{array}$$

Thus, the last stage requires 7 flops. Therefore, the total number of operations performed is

$$14 + 11 + 7 = 32$$

flops. As already indicated by the theoretical analysis, this is too much work and this is why in Numerical Analysis we prefer a more efficient approach.

In this setting, meaning that we are going to apply the Gaussian elimination method (GEM, for short), it is enough for us to reduce the matrix to upper triangular form; there is no need to continue all the way to the identity matrix on the left-hand side the way we did using the Gauss-Jordan method. Moreover, some of the operations used in the Gauss-Jordan method are not allowed in Numerical Analysis. In particular, we do not allow multiplying or dividing an entire row by a scalar. Therefore, an operation such as $-\frac{1}{2} \cdot (\#3) \to (\#3)$ is not admissible.

We now solve the same system by Gaussian elimination, using only the admissible operations. Notice first that in the first stage of the previous method we already have used only admissible operations. That is, after doing so, we have the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 2 & -2 \end{array}\right)$$

and we have done 14 operations. In the second stage, we perform only one more operation given by

$$(\#3) - 2 \cdot (\#2) \ \to \ (\#3).$$

Hence,

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 2 & -2 \end{array}\right) \xrightarrow{(\#3) - 2 \cdot (\#2) \to (\#3)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{array}\right).$$

In the second stage, we have spent $1 + 2 \cdot 2 = 5$ more operations. Therefore, the total of operations is 15 + 5 = 19 operations. We have therefore reached an upper triangular system given by

$$\begin{cases} x + y - 2z = 3, \\ 2y + 2z = -2, \\ -2z = 2. \end{cases}$$

We now solve by back substitution (BS, for short). From the third equation, $z = -1$ (which gives 1 more operation). Substituting into the second equation, we obtain

$$y = \frac{1}{2} \cdot (-2 - 2z) = 0$$

(which gives 3 more operations. Finally, substituting into the first equation gives

$$x = 3 - y + 2z = 1$$

(which gives 3 more operations). Thus, the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

with a total of 7 operations. The total of operations for the whole procedure is

$$19 + 7 = 26$$

operations.

33.3 Problem 2

We write the system in augmented matrix form:

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 4 & 4 & 6 & 2 \\ 2 & -5 & -3 & 0 \end{array}\right).$$

We now apply Gaussian elimination in the same style as before. In the first stage, we use the pivot in the first row and perform the operations

$$\begin{aligned} (\#2) - 2\cdot(\#1) &\;\to\; (\#2), \\ (\#3) - (\#1) &\;\to\; (\#3). \end{aligned}$$

Thus,

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 4 & 4 & 6 & 2 \\ 2 & -5 & -3 & 0 \end{array}\right) \overset{(\#2)-2\cdot(\#1)\to(\#2)}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & -5 & -3 & 0 \end{array}\right)$$

$$\overset{(\#3)-(\#1)\to(\#3)}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -6 & -5 & -1 \end{array}\right).$$

In the second stage, we eliminate the entry below the second pivot:

$$(\#3) + 3\cdot(\#2) \;\to\; (\#3).$$

Hence,

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & -6 & -5 & -1 \end{array}\right) \overset{(\#3)+3\cdot(\#2)\to(\#3)}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right).$$

This gives a total of

$$(1 + 3\cdot 2) + (1 + 3\cdot 1) + (1 + 2\cdot 2) = 7 + 4 + 5 = 16$$

operations in total. We have therefore obtained the upper triangular system

$$\begin{cases} 2x + y + 2z = 1, \\ 2y + 2z = 0, \\ z = -1. \end{cases}$$

We now solve by back substitution. From the third equation, $z = -1$. Substituting into the second equation, we get that $y = -z$ and then $y = 1$. Then,

$$x = \frac{1}{2}(1 - y - 2z) = 1$$

and this gives a total of 6 operations more. Thus, the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

and a total of 22 flops to reach the solution of the system.