Midterm
2025-2026 - 2nd semester
Differential Equations and Numerical Methods
(BE5B01DEN)
Prague, Dejvice, April 21st, 2026
**Variant 1**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = y^2, \\ y(0) = -1. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' + 2y' = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' + 2y' = e^x + 7x + 1.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Using the matrix approach, find a general solution of the system

$$\begin{cases} y_1' = -2y_1 - y_2, \\ y_2' = -y_1 - 2y_2. \end{cases}$$

*Solutions for Variant 1*

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int dx \quad \Longrightarrow \quad -\frac{1}{y} = x + C.$$

Hence,

$$y = -\frac{1}{x+C}.$$

Now we impose the initial condition $y(0) = -1$ and find that

$$-1 = -\frac{1}{0+C}.$$

Therefore, $C = 1$. Hence, the solution of the initial value problem is

$$y(x) = -\frac{1}{x+1}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' + 2y' = 0$. Its characteristic equation is $\lambda^2 + 2\lambda = 0$. Factoring, we obtain $\lambda(\lambda + 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = 0$ and $\lambda_2 = -2$. Hence, the general solution is given by

$$y(x) = C_1 + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 + C_2 e^{-2x}$. Since $e^{-2x} \to 0$ as $x \to +\infty$, every solution satisfies $y(x) \to C_1$ as $x \to +\infty$. Thus, every solution tends to a finite constant at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 + C_2 e^{-2x}$, we can compute $y'(x) = -2C_2 e^{-2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ -2C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = -\frac{1}{2}, \qquad C_2 = -\frac{1}{2}.$$

Therefore, the required solution is

$$y(x) = -\frac{1}{2} - \frac{1}{2}e^{-2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' + 2y' = e^{1x} + 7x + 1$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^x$, since 1 is not a root of the characteristic polynomial, we guess a particular solution of the form $Ae^x$.
- ⋆ For the polynomial term of degree one given by $7x + 1$, we would normally guess a polynomial of degree 1, namely $Bx + C$. However, since 0 is a root of the characteristic polynomial, we must correct this guess and multiply it by $x$. Therefore, for this part we guess $x(Bx + C) = Bx^2 + Cx$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = Ae^x + Bx^2 + Cx.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & -1 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3.$$

Thus, $\lambda^2 + 4\lambda + 3 = 0$, that is, $(\lambda + 3)(\lambda + 1) = 0$. Therefore, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. For $\lambda_1 = -3$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = -1$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{-3x}\vec{v}_1 + C_2 e^{-x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{-3x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{-3x} + C_2 e^{-x}$ and $y_2(x) = C_1 e^{-3x} - C_2 e^{-x}$ for every $x \in \mathbb{R}$.

Midterm
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**Variant 2**

**Surname/Name:** ______________________________________________
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- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = 2y^2, \\ y(0) = -\frac{1}{2}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' - y' - 2y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' - y' - 2y = e^{2x} + 8x - 2.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = -2y_1 + 4y_2, \\ y_2' = 2y_2. \end{cases}$$

*Solutions for Variant 2*

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = 2y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 2y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = 2\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int 2\,dx \quad \Longrightarrow \quad -\frac{1}{y} = 2x + C.$$

Hence,

$$y = -\frac{1}{2x+C}.$$

Now we impose the initial condition $y(0) = -\frac{1}{2}$ and find that

$$-\frac{1}{2} = -\frac{1}{0+C}.$$

Therefore, $C = 2$. Hence, the solution of the initial value problem is

$$y(x) = -\frac{1}{2x+2}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' - y' - 2y = 0$. Its characteristic equation is $\lambda^2 - \lambda - 2 = 0$. Factoring, we obtain $(\lambda - 2)(\lambda + 1) = 0$. Therefore, the characteristic roots are $\lambda_1 = 2$ and $\lambda_2 = -1$. Hence, the general solution is given by

$$y(x) = C_1 e^{2x} + C_2 e^{-x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{2x} + C_2 e^{-x}$. Since $e^{-x} \to 0$ but $e^{2x}$ grows exponentially as $x \to +\infty$, every solution with $C_1 \neq 0$ is unbounded at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{2x} + C_2 e^{-x}$, we can compute $y'(x) = 2C_1 e^{2x} - 1C_2 e^{-x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ 2C_1 - 1C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = 0, \qquad C_2 = -1.$$

Therefore, the required solution is

$$y(x) = -e^{-x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' - y' - 2y = e^{2x} + 8x - 2$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{2x}$, since 2 is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^{2x}$.
- ⋆ For the polynomial term of degree one given by $8x - 2$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx + C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^{2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -2 & 4 \\ 0 & 2 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 4 \\ 0 & 2-\lambda \end{vmatrix} = \lambda^2 - 0\lambda - 4.$$

Thus, $\lambda^2 - 0\lambda - 4 = 0$, that is, $(\lambda + 2)(\lambda - 2) = 0$. Therefore, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 2$. For $\lambda_1 = -2$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 2$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{-2x} \vec{v}_1 + C_2 e^{2x} \vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{-2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{-2x} + C_2 e^{2x}$ and $y_2(x) = C_2 e^{2x}$ for every $x \in \mathbb{R}$.

Midterm
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**Variant 3**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = 3y^2, \\ y(0) = 1. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' - 4y' + 4y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' - 4y' + 4y = e^{2x} + 10x + 1.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = -y_1, \\ y_2' = -4y_1 + 3y_2. \end{cases}$$

*Solutions for Variant 3*

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = 3y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 3y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = 3\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int 3\,dx \quad \Longrightarrow \quad -\frac{1}{y} = 3x + C.$$

Hence,

$$y = -\frac{1}{3x + C}.$$

Now we impose the initial condition $y(0) = 1$ and find that

$$1 = -\frac{1}{0 + C}.$$

Therefore, $C = -1$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{1 - 3x}.$$

The maximal interval containing 0 on which this solution is defined is $(-\infty, \frac{1}{3})$.

**2.** a) We consider the homogeneous linear equation $y'' - 4y' + 4y = 0$. Its characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$. Factoring, we obtain $(\lambda - 2)(\lambda - 2) = 0$. Therefore, the characteristic root is $\lambda = 2$ with multiplicity two. Hence, the general solution is given by

$$y(x) = C_1 e^{2x} + C_2 x e^{2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{2x} + C_2 x e^{2x}$. Since $e^{2x}$ grows exponentially as $x \to +\infty$, every non-zero solution is unbounded at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{2x} + C_2 x e^{2x}$, we can compute $y'(x) = 2C_1 e^{2x} + C_2 e^{2x} + 2C_2 x e^{2x}$. Using $y(0) = -1$, we get

$$C_1 = -1.$$

Using $y'(0) = 1$, we obtain

$$C_2 + 2C_1 = 1 \quad \Longrightarrow \quad C_2 + 2\,(-1) = 1 \quad \Longrightarrow \quad C_2 = 3.$$

Therefore, the required solution is

$$y(x) = 3xe^{2x} - e^{2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' - 4y' + 4y = e^{2x} + 10x + 1$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{2x}$, since 2 is a double root of the characteristic polynomial, we must multiply the usual guess by $x^2$. Therefore, for this part we guess $x^2Ae^{2x}$.
- ⋆ For the polynomial term of degree one given by $10x + 1$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = x^2Ae^{2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -1 & 0 \\ -4 & 3 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 \\ -4 & 3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3.$$

Thus, $\lambda^2 - 2\lambda - 3 = 0$, that is, $(\lambda + 1)(\lambda - 3) = 0$. Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$. For $\lambda_1 = -1$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 0 & 0 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 3$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -4 & 0 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1e^{-x}\vec{v}_1 + C_2e^{3x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1e^{-x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2e^{3x}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1e^{-x}$ and $y_2(x) = C_1e^{-x} + C_2e^{3x}$ for every $x \in \mathbb{R}$.

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**Variant 4**

**Surname/Name:** ______________________________________________
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- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = -y^2, \\ y(0) = 1. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' + 4y' + 4y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' + 4y' + 4y = e^{-2x} + 12x + 4.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = 5y_1 - 4y_2, \\ y_2' = 2y_1 - y_2. \end{cases}$$

*Solutions for Variant 4*

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = -y^2 \implies \frac{dy}{dx} = -y^2 \implies \frac{dy}{y^2} = -dx.$$

Integrating both sides, we get

$$\int y^{-2}\, dy = -\int dx \implies -\frac{1}{y} = -x + C.$$

Hence,

$$y = -\frac{1}{C - x}.$$

Now we impose the initial condition $y(0) = 1$ and find that

$$1 = -\frac{1}{0 + C}.$$

Therefore, $C = -1$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{x+1}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' + 4y' + 4y = 0$. Its characteristic equation is $\lambda^2 + 4\lambda + 4 = 0$. Factoring, we obtain $(\lambda+2)(\lambda+2) = 0$. Therefore, the characteristic root is $\lambda = -2$ with multiplicity two. Hence, the general solution is given by

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$. Since $e^{-2x} \to 0$ as $x \to +\infty$, and the exponential decay dominates the linear factor, every solution satisfies $y(x) \to 0$ as $x \to +\infty$.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$, we can compute $y'(x) = -2C_1 e^{-2x} C_2 e^{-2x} - 2C_2 x e^{-2x}$. Using $y(0) = -1$, we get

$$C_1 = -1.$$

Using $y'(0) = 1$, we obtain

$$C_2 + -2C_1 = 1 \implies C_2 + -2\,(-1) = 1 \implies C_2 = -1.$$

Therefore, the required solution is

$$y(x) = -e^{-2x} - x e^{-2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' + 4y' + 4y = e^{-2x} + 12x + 4$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{-2x}$, since $-2$ is a double root of the characteristic polynomial, we must multiply the usual guess by $x^2$. Therefore, for this part we guess $x^2Ae^{-2x}$.
- ⋆ For the polynomial term of degree one given by $12x + 4$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = x^2Ae^{-2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3.$$

Thus, $\lambda^2 - 4\lambda + 3 = 0$, that is, $(\lambda - 1)(\lambda - 3) = 0$. Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. For $\lambda_1 = 1$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 3$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1e^x\vec{v}_1 + C_2e^{3x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1e^x + 2C_2e^{3x}$ and $y_2(x) = C_1e^x + C_2e^{3x}$ for every $x \in \mathbb{R}$.

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**Variant 5**

**Surname/Name:** ______________________________________________
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- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = 4y^2, \\ y(0) = -\frac{1}{4}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' - 3y' + 2y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' - 3y' + 2y = e^{2x} + 14x - 3.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = 3y_1 - y_2, \\ y_2' = -y_1 + 3y_2. \end{cases}$$

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = 4y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 4y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = 4\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int 4\,dx \quad \Longrightarrow \quad -\frac{1}{y} = 4x + C.$$

Hence,

$$y = -\frac{1}{4x + C}.$$

Now we impose the initial condition $y(0) = -\frac{1}{4}$ and find that

$$-\frac{1}{4} = -\frac{1}{0 + C}.$$

Therefore, $C = 4$. Hence, the solution of the initial value problem is

$$y(x) = -\frac{1}{4x + 4}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' - 3y' + 2y = 0$. Its characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$. Factoring, we obtain $(\lambda - 1)(\lambda - 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = 1$ and $\lambda_2 = 2$. Hence, the general solution is given by

$$y(x) = C_1 e^{x} + C_2 e^{2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{1x} + C_2 e^{2x}$. Since both exponential terms grow as $x \to +\infty$, every non-zero solution is unbounded at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{x} + C_2 e^{2x}$, we can compute $y'(x) = 1C_1 e^{x} + 2C_2 e^{2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ 1C_1 + 2C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = -3, \qquad C_2 = 2.$$

Therefore, the required solution is

$$y(x) = -3e^{x} + 2e^{2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' - 3y' + 2y = e^{2x} + 14x - 3$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{2x}$, since 2 is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^{2x}$.
- ⋆ For the polynomial term of degree one given by $14x - 3$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^{2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8.$$

Thus, $\lambda^2 - 6\lambda + 8 = 0$, that is, $(\lambda - 2)(\lambda - 4) = 0$. Therefore, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$. For $\lambda_1 = 2$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 4$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{2x}\vec{v}_1 + C_2 e^{4x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{4x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{2x} + C_2 e^{4x}$ and $y_2(x) = C_1 e^{2x} - C_2 e^{4x}$ for every $x \in \mathbb{R}$.

Midterm
2025-2026 - 2nd semester
Differential Equations and Numerical Methods
(BE5B01DEN)
Prague, Dejvice, April 21st, 2026
**Variant 6**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = 2y^2, \\ y(0) = \frac{1}{2}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' + 3y' + 2y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' + 3y' + 2y = e^{-x} + 16x - 1.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = -4y_1 + 4y_2, \\ y_2' = 0. \end{cases}$$

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = 2y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 2y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = 2\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int 2\,dx \quad \Longrightarrow \quad -\frac{1}{y} = 2x + C.$$

Hence,

$$y = -\frac{1}{2x + C}.$$

Now we impose the initial condition $y(0) = \frac{1}{2}$ and find that

$$\frac{1}{2} = -\frac{1}{0 + C}.$$

Therefore, $C = -2$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{2 - 2x}.$$

The maximal interval containing 0 on which this solution is defined is $(-\infty, 1)$.

**2.** a) We consider the homogeneous linear equation $y'' + 3y' + 2y = 0$. Its characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$. Factoring, we obtain $(\lambda + 1)(\lambda + 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence, the general solution is given by

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{-x} + C_2 e^{-2x}$. Since both exponential terms decay to 0 as $x \to +\infty$, every solution satisfies $y(x) \to 0$ as $x \to +\infty$.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{-x} + C_2 e^{-2x}$, we can compute $y'(x) = -1C_1 e^{-x} - 2C_2 e^{-2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ -C_1 - 2C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = -1, \qquad C_2 = 0.$$

Therefore, the required solution is

$$y(x) = -e^{-x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' + 3y' + 2y = e^{-1x} + 16x - 1$. We treat the right-hand side term by term and then apply the Superposition Principle.

- $\star$ For the term $e^{-x}$, since $-1$ is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^{-x}$.
- $\star$ For the polynomial term of degree one given by $16x - 1$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^{-x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 4 \\ 0 & 0-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 0.$$

Thus, $\lambda^2 + 4\lambda + 0 = 0$, that is, $(\lambda + 4)(\lambda - 0) = 0$. Therefore, the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 0$. For $\lambda_1 = -4$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 0 & 4 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 0$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{-4x}\vec{v}_1 + C_2\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{-4x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{-4x} + C_2$ and $y_2(x) = C_2$ for every $x \in \mathbb{R}$.

Midterm
2025-2026 - 2nd semester
Differential Equations and Numerical Methods
(BE5B01DEN)
Prague, Dejvice, April 21st, 2026
**Variant 7**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = -3y^2, \\ y(0) = \frac{1}{3}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' - 2y' = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' - 2y' = e^x + 5x - 2.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = -y_1, \\ y_2' = -2y_1 + y_2. \end{cases}$$

*Solutions for Variant 7*

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = -3y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = -3y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = -3\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int -3\,dx \quad \Longrightarrow \quad -\frac{1}{y} = -3x + C.$$

Hence,

$$y = -\frac{1}{-3x + C}.$$

Now we impose the initial condition $y(0) = \frac{1}{3}$ and find that

$$\frac{1}{3} = -\frac{1}{0 + C}.$$

Therefore, $C = -3$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{3x + 3}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' - 2y' = 0$. Its characteristic equation is $\lambda^2 - 2\lambda = 0$. Factoring, we obtain $\lambda(\lambda - 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = 0$ and $\lambda_2 = 2$. Hence, the general solution is given by

$$y(x) = C_1 + C_2 e^{2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 + C_2 e^{2x}$. Since $e^{2x} \to +\infty$ as $x \to +\infty$, every solution with $C_2 \neq 0$ is unbounded at infinity. The only solutions that remain bounded are the constant ones, obtained when $C_2 = 0$.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 + C_2 e^{2x}$, we can compute $y'(x) = 2C_2 e^{2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ 2C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = -\frac{3}{2}, \qquad C_2 = \frac{1}{2}.$$

Therefore, the required solution is

$$y(x) = -\frac{3}{2} + \frac{1}{2}e^{2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y'' - 2y' = e^{1x} + 5x - 2$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^x$, since 1 is not a root of the characteristic polynomial, we guess a particular solution of the form $Ae^x$.
- ⋆ For the polynomial term of degree one given by $5x - 2$, we would normally guess a polynomial of degree 1, namely $Bx + C$. However, since 0 is a root of the characteristic polynomial, we must correct this guess and multiply it by $x$. Therefore, for this part we guess $x(Bx + C) = Bx^2 + Cx$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = Ae^x + Bx^2 + Cx.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 0\lambda - 1.$$

Thus, $\lambda^2 - 0\lambda - 1 = 0$, that is, $(\lambda + 1)(\lambda - 1) = 0$. Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$. For $\lambda_1 = -1$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 1$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -2 & 0 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{-x} \vec{v}_1 + C_2 e^x \vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{-x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^x \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{-x}$ and $y_2(x) = C_1 e^{-x} + C_2 e^x$ for every $x \in \mathbb{R}$.

Midterm
2025-2026 - 2nd semester
Differential Equations and Numerical Methods
(BE5B01DEN)
Prague, Dejvice, April 21st, 2026
**Variant 8**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = 5y^2, \\ y(0) = -\frac{1}{5}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' + y' - 2y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 1$.

**3.** Consider the equation

$$y'' + y' - 2y = e^x + 6x + 3.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = 9y_1 - 8y_2, \\ y_2' = 4y_1 - 3y_2. \end{cases}$$

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = 5y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = 5y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = 5\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int 5\,dx \quad \Longrightarrow \quad -\frac{1}{y} = 5x + C.$$

Hence,

$$y = -\frac{1}{5x + C}.$$

Now we impose the initial condition $y(0) = -\frac{1}{5}$ and find that

$$-\frac{1}{5} = -\frac{1}{0 + C}.$$

Therefore, $C = 5$. Hence, the solution of the initial value problem is

$$y(x) = -\frac{1}{5x + 5}.$$

The maximal interval containing 0 on which this solution is defined is $(-1, +\infty)$.

**2.** a) We consider the homogeneous linear equation $y'' + y' - 2y = 0$. Its characteristic equation is $\lambda^2 + \lambda - 2 = 0$. Factoring, we obtain $(\lambda - 1)(\lambda + 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = 1$ and $\lambda_2 = -2$. Hence, the general solution is given by

$$y(x) = C_1 e^{x} + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{x} + C_2 e^{-2x}$. Since $e^{-2x} \to 0$ but $e^{x}$ grows exponentially as $x \to +\infty$, every solution with $C_1 \neq 0$ is unbounded at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 1$. From the general solution, $y(x) = C_1 e^{x} + C_2 e^{-2x}$, we can compute $y'(x) = 1C_1 e^{x} - 2C_2 e^{-2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ 1C_1 - 2C_2 = 1. \end{cases}$$

Solving this system, we find

$$C_1 = -\frac{1}{3}, \qquad C_2 = -\frac{2}{3}.$$

Therefore, the required solution is

$$y(x) = -\frac{1}{3}e^{x} - \frac{2}{3}e^{-2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y''+y'-2y = e^x+6x+3$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^x$, since 1 is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^x$.
- ⋆ For the polynomial term of degree one given by $6x+3$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^x + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} 9 & -8 \\ 4 & -3 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A-\lambda I) = \begin{vmatrix} 9-\lambda & -8 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5.$$

Thus, $\lambda^2 - 6\lambda + 5 = 0$, that is, $(\lambda-1)(\lambda-5) = 0$. Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$. For $\lambda_1 = 1$, we solve $(A-\lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 8 & -8 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 5$, we solve $(A-\lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 4 & -8 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^x \vec{v}_1 + C_2 e^{5x} \vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{5x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^x + 2C_2 e^{5x}$ and $y_2(x) = C_1 e^x + C_2 e^{5x}$ for every $x \in \mathbb{R}$.

Midterm

2025-2026 - 2nd semester

Differential Equations and Numerical Methods

(BE5B01DEN)

Prague, Dejvice, April 21st, 2026

**Variant 9**

**Surname/Name:** ______________________________________________

(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = y^2, \\ y(0) = \frac{1}{2}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' - y' - 6y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = -1$ and $y'(0) = 2$.

**3.** Consider the equation

$$y'' - y' - 6y = e^{-2x} + 8x - 2.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = y_1 - y_2, \\ y_2' = -y_1 + y_2. \end{cases}$$

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = dx.$$

Integrating both sides, we get

$$\int y^{-2}\, dy = \int dx \quad \Longrightarrow \quad -\frac{1}{y} = x + C.$$

Hence,

$$y = -\frac{1}{x+C}.$$

Now we impose the initial condition $y(0) = \frac{1}{2}$ and find that

$$\frac{1}{2} = -\frac{1}{0+C}.$$

Therefore, $C = -2$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{2-x}.$$

The maximal interval containing 0 on which this solution is defined is $(-\infty, 2)$.

**2.** a) We consider the homogeneous linear equation $y'' - y' - 6y = 0$. Its characteristic equation is $\lambda^2 - \lambda - 6 = 0$. Factoring, we obtain $(\lambda - 3)(\lambda + 2) = 0$. Therefore, the characteristic roots are $\lambda_1 = 3$ and $\lambda_2 = -2$. Hence, the general solution is given by

$$y(x) = C_1 e^{3x} + C_2 e^{-2x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{3x} + C_2 e^{-2x}$. Since $e^{-2x} \to 0$ but $e^{3x}$ grows exponentially as $x \to +\infty$, every solution with $C_1 \neq 0$ is unbounded at infinity.

c) We now impose the initial conditions $y(0) = -1$ and $y'(0) = 2$. From the general solution, $y(x) = C_1 e^{3x} + C_2 e^{-2x}$, we can compute $y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = -1, \\ 3C_1 - 2C_2 = 2. \end{cases}$$

Solving this system, we find

$$C_1 = 0, \qquad C_2 = -1.$$

Therefore, the required solution is

$$y(x) = -e^{-2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y''-y'-6y = e^{-2x}+8x-2$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{-2x}$, since $-2$ is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^{-2x}$.
- ⋆ For the polynomial term of degree one given by $8x-2$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^{-2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 0.$$

Thus, $\lambda^2 - 2\lambda + 0 = 0$, that is, $(\lambda - 0)(\lambda - 2) = 0$. Therefore, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. For $\lambda_1 = 0$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2 = 2$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1\vec{v}_1 + C_2 e^{2x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 + C_2 e^{2x}$ and $y_2(x) = C_1 - C_2 e^{2x}$ for every $x \in \mathbb{R}$.

Midterm
2025-2026 - 2nd semester
Differential Equations and Numerical Methods
(BE5B01DEN)
Prague, Dejvice, April 21st, 2026
**Variant 10**

**Surname/Name:** ______________________________________________
(in CAPITAL LETTERS)

- ⋆ The test is worth **20 points** in total and **10 points** are required for *zápočet*.
- ⋆ Calculators are **not** allowed.
- ⋆ Each problem is worth **5 points**.

**1.** Find the solution of the initial value problem

$$\begin{cases} y' = -2y^2, \\ y(0) = -\frac{1}{2}. \end{cases}$$

**2.** a) Find the general solution of the equation $y'' + y' - 6y = 0$.

b) Discuss its typical behaviour at infinity.

c) Find the solution given by the initial conditions $y(0) = 1$ and $y'(0) = 2$.

**3.** Consider the equation

$$y'' + y' - 6y = e^{2x} + 10x + 1.$$

Guess the general form (do **not** find the constants) of a particular solution $y_p$.

**4.** Find a general solution of the system

$$\begin{cases} y_1' = -2y_1 + 6y_2, \\ y_2' = 4y_2. \end{cases}$$

**1.** First notice that this is a separable equation. So, for $y \neq 0$, we have that

$$y' = -2y^2 \quad \Longrightarrow \quad \frac{dy}{dx} = -2y^2 \quad \Longrightarrow \quad \frac{dy}{y^2} = -2\,dx.$$

Integrating both sides, we get

$$\int y^{-2}\,dy = \int -2\,dx \quad \Longrightarrow \quad -\frac{1}{y} = -2x + C.$$

Hence,

$$y = -\frac{1}{-2x + C}.$$

Now we impose the initial condition $y(0) = -\frac{1}{2}$ and find that

$$-\frac{1}{2} = -\frac{1}{0 + C}.$$

Therefore, $C = 2$. Hence, the solution of the initial value problem is

$$y(x) = \frac{1}{2x - 2}.$$

The maximal interval containing 0 on which this solution is defined is $(-\infty, 1)$.

**2.** a) We consider the homogeneous linear equation $y'' + y' - 6y = 0$. Its characteristic equation is $\lambda^2 + \lambda - 6 = 0$. Factoring, we obtain $(\lambda - 2)(\lambda + 3) = 0$. Therefore, the characteristic roots are $\lambda_1 = 2$ and $\lambda_2 = -3$. Hence, the general solution is given by

$$y(x) = C_1 e^{2x} + C_2 e^{-3x}$$

for every $x \in \mathbb{R}$.

b) As we have seen before, a typical solution has the form $y(x) = C_1 e^{2x} + C_2 e^{-3x}$. Since $e^{-3x} \to 0$ but $e^{2x}$ grows exponentially as $x \to +\infty$, every solution with $C_1 \neq 0$ is unbounded at infinity.

c) We now impose the initial conditions $y(0) = 1$ and $y'(0) = 2$. From the general solution, $y(x) = C_1 e^{2x} + C_2 e^{-3x}$, we can compute $y'(x) = 2C_1 e^{2x} - 3C_2 e^{-3x}$. Using the initial conditions, we obtain

$$\begin{cases} C_1 + C_2 = 1, \\ 2C_1 - 3C_2 = 2. \end{cases}$$

Solving this system, we find

$$C_1 = 1, \qquad C_2 = 0.$$

Therefore, the required solution is

$$y(x) = e^{2x}, \quad x \in \mathbb{R}.$$

**3.** We want to guess the general form of a particular solution for $y''+y'-6y = e^{2x}+10x+1$. We treat the right-hand side term by term and then apply the Superposition Principle.

- ⋆ For the term $e^{2x}$, since 2 is a root of the characteristic polynomial, we must correct the usual guess and multiply it by $x$. Therefore, for this part we guess $xAe^{2x}$.
- ⋆ For the polynomial term of degree one given by $10x+1$, since 0 is not a root of the characteristic polynomial, we guess a polynomial of degree 1, namely $Bx+C$.

Hence, by the Superposition Principle, the general form of a particular solution is

$$y_p(x) = xAe^{2x} + Bx + C.$$

**4.** The matrix of this system is given by

$$A = \begin{pmatrix} -2 & 6 \\ 0 & 4 \end{pmatrix}.$$

Next, we find the eigenvalues from the characteristic equation as follows

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 6 \\ 0 & 4-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 8.$$

Thus, $\lambda^2 - 2\lambda - 8 = 0$, that is, $(\lambda+2)(\lambda-4) = 0$. Therefore, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 4$. For $\lambda_1 = -2$, we solve $(A - \lambda_1 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} 0 & 6 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda_2 = 4$, we solve $(A - \lambda_2 I)\vec{v} = 0$, that is,

$$\begin{pmatrix} -6 & 6 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One eigenvector is

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the general solution of the system is

$$\vec{y}(x) = C_1 e^{-2x}\vec{v}_1 + C_2 e^{4x}\vec{v}_2.$$

That is,

$$\vec{y}(x) = C_1 e^{-2x} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{4x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the solution of the system of ODEs is given by $y_1(x) = C_1 e^{-2x} + C_2 e^{4x}$ and $y_2(x) = C_2 e^{4x}$ for every $x \in \mathbb{R}$.