29 Lecture #20: Wednesday, April 22nd, 2026

29.1 Stability of solutions

A general first-order system of ordinary differential equations can be written in the form

$$\begin{cases} y_1' = F_1(t, y_1, y_2, \ldots, y_n), \\ y_2' = F_2(t, y_1, y_2, \ldots, y_n), \\ \quad \vdots \\ y_n' = F_n(t, y_1, y_2, \ldots, y_n). \end{cases} \tag{30}$$

Here, t usually represents time, while the unknown functions $y_1, \ldots, y_n$ describe the evolution of the system. In many real-life applications, the functions $F_1, \ldots, F_n$ are nonlinear, and in that case it is often impossible to obtain an explicit analytic solution. For this reason, one is naturally led to the use of numerical methods, and this is precisely where numerical analysis plays a fundamental role. We shall analyze the solutions of the system in order to understand their qualitative behavior, predict their long-term evolution and determine toward which states the system tends.

Example 29.1. An example of a non-linear system is

$$\begin{cases} y_1' = -2y_2^2, \\ y_2' = -\dfrac{y_1}{y_2}. \end{cases} \tag{31}$$

The functions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$ for $t \geqslant 0$ are solutions for this system. At this stage, it is not important how these expressions were obtained. What matters for the moment is to visualize their behavior. We observe that both solutions tend to 0 as $t \to \infty$, but y_1 approaches 0 faster than y_2. This can be clearly seen in Figure 118.

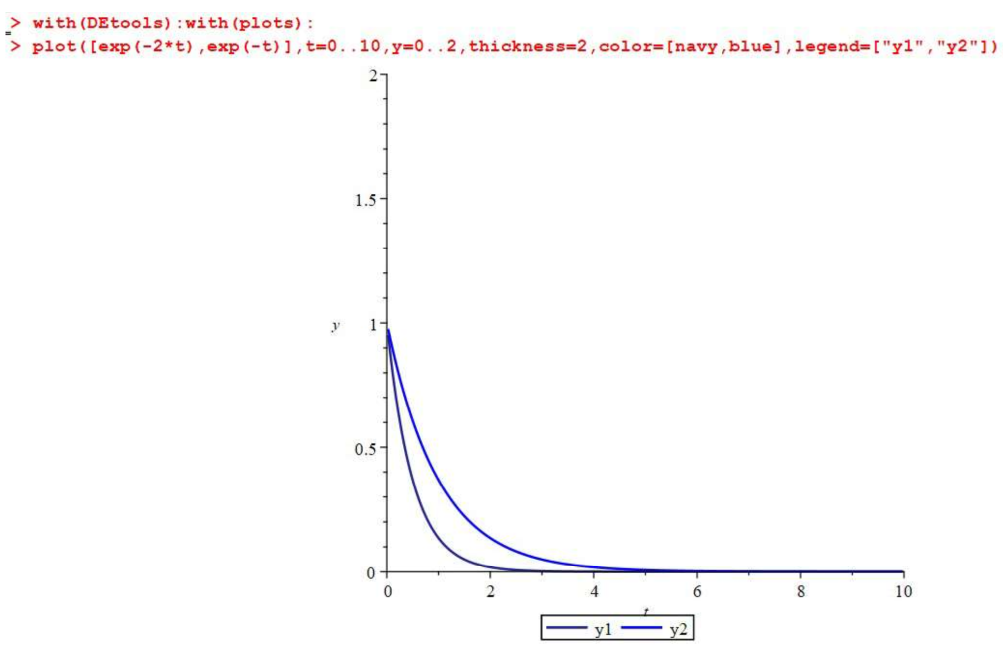

Figure 118: Graphs of the solutions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$.

It can be helpful to look at a graph such as this in order to gain some intuition about the solutions of a given system. However, this is not always the most convenient point of view, since in some cases the picture may become rather complicated, as illustrated in Figure 119. Indeed, in this plot we see five different solutions, that is, ten functions in total.

```
> plot([exp(-2*t),exp(-t),10*exp(-2*t),-10*exp(-t),3*exp(-2*t)*(2-0.01*exp(4*t)),3*exp(-t)*sqrt(2+0.01*exp(4*
  t)),6*exp(-2*t)*(0.1-0.3*exp(4*t)),6*exp(-t)*sqrt(0.1+0.3*exp(4*t)),2*exp(-2*t)*(4-(-0.0001)*exp(4*t)),-2*
  exp(-t)*sqrt(4+(-0.0001)*exp(4*t))],t=0..6,y=-10..10,thickness=2,color=[navy,blue,sienna,gold,aquamarine,
  green,red,coral,magenta,plum]);
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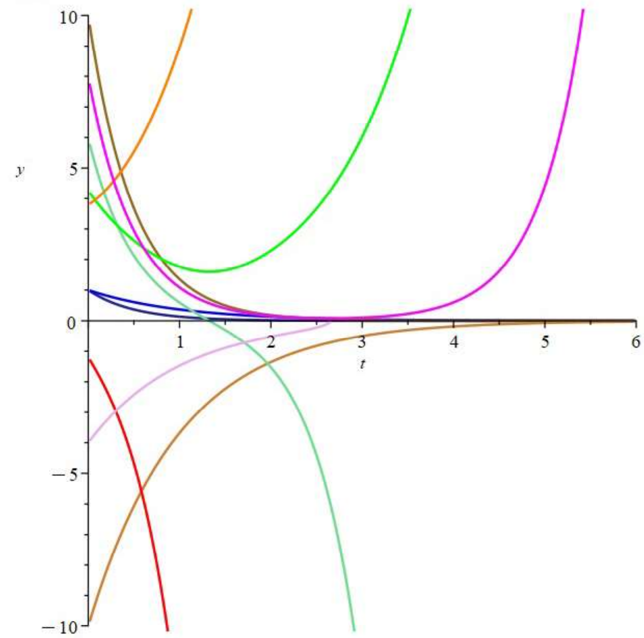

Figure 119: Several different solutions for a large system of ODEs.

As we have seen before, the system (30) can be written in vector form as

$$\vec{y}\,' = F(t, \vec{y}).$$

Notice that, if we remove the arrows, we recover the scalar equation $y' = F(t, y)$, which is precisely the general form of a first-order ordinary differential equation studied earlier. As a vector function has only one graph, this might simplify our previous example.

Example 29.2. Let us go back to the system (31) and let us consider the graph in the vector form as given in Figure 120. This plot represents the parametric curve $(y_1(t), t, y_2(t)) = (e^{-2t}, t, e^{-t})$ for $t \in [0, 10]$. Thus, it shows the evolution of the solution in three-dimensional space, with time as one of the axes. At $t = 0$, the solution starts at $(y_1(0), t, y_2(0)) = (1, 0, 1)$. As t increases, both $y_1(t)$ and $y_2(t)$ decrease to 0, so the curve moves forward in the t-direction while approaching the t-axis in the (y_1, y_2)-components. The graph also makes clear that $y_1(t) = e^{-2t}$ tends to 0 faster than $y_2(t) = e^{-t}$. Geometrically, this means that the y_1-coordinate decreases more rapidly, so the curve approaches the plane $y_1 = 0$ sooner than it approaches the plane $y_2 = 0$. Another important feature is that the curve is smooth and never goes backward in time, since t increases monotonically. This reflects the continuous evolution of the solution from the initial condition. Moreover, both components remain positive for every $t \geqslant 0$. Hence, the trajectory stays in the region where $y_1 > 0$ and $y_2 > 0$, while getting closer and closer to the line $\{(0, t, 0) : t \geqslant 0\}$. From a qualitative point of view, the figure shows exponential decay in both variables, but with different rates.

```
> spacecurve([exp(-2*t),t,exp(-t)],t=0..10,thickness=3,color=navy,view=[0..2,0..10,0..2],axes=normal,labels=
  ["y1","t","y2"]);
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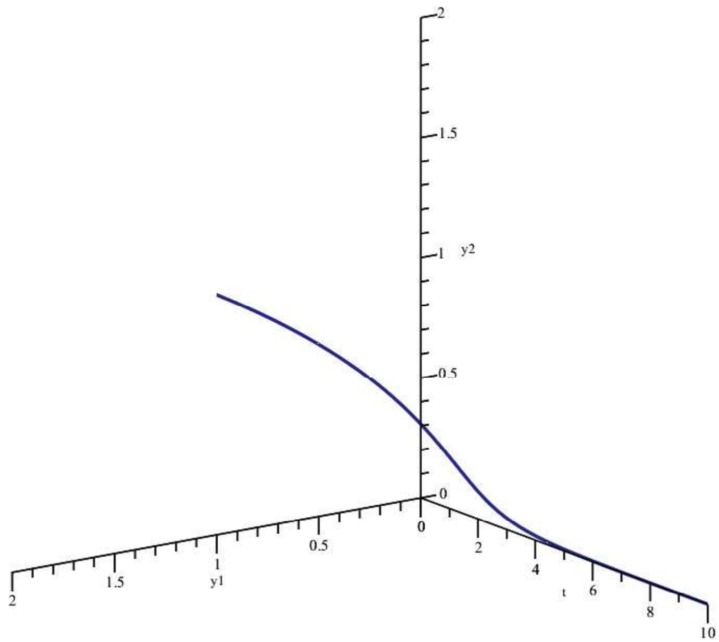

Figure 120: System (31) in a vector form.

This approach also has several disadvantages, since in practice it usually requires the use of a computer. Moreover, if we consider a 3×3 system, then incorporating time would lead to a four-dimensional representation, which is no longer easy to visualize. For this reason, we shall restrict ourselves for the moment to 2×2 systems.

Let us analyze now the stationary solutions. They are of the form $\vec{y}(t) = \vec{y}_0 \in \mathbb{R}^n$ and geometrically they can be visualized as in Figure 121.

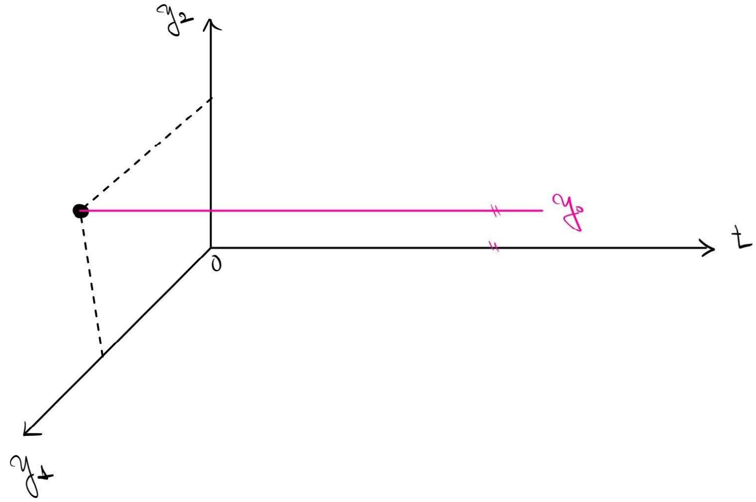

Figure 121: Vector form of a stationary solution $\vec{y}_0$.

The value $\vec{y}_0$ is called an **equilibrium**, as we have seen before. How can we recognize stationary solutions? Well, they are constant, and then their derivative must be zero; that is, we are looking for $\vec{y}_0 = (y_{01}, \dots, y_{0n}) \in \mathbb{R}^n$ to be such that $\vec{F}(t, \vec{y}_0) = 0$. This means that

$$F_i(t, y_{01}, \dots, y_{0n}) = 0$$

for every $i = 1, \dots, n$ and every $t \in \mathbb{R}$.

Let us go back to the system (31) and ask ourselves if there are any stationary solutions. First of all, y_2 must be different from 0 in order for the second equation to make sense. On the other hand, for a stationary solution we must have

$$-\frac{y_1}{y_2} = 0,$$

and therefore $y_1 = 0$. Substituting this into the first equation, we obtain $-2y_2^2 = 0$, which implies $y_2 = 0$. This is impossible, since the second equation is not defined at $y_2 = 0$. Hence, we have no stationary solutions in this case.

Example 29.3. Let us analyze the system

$$\begin{cases} y_1' = y_2^2 - 1, \\ y_2' = y_2 - y_1. \end{cases}$$

We want to find its stationary solutions. To do so, we impose $y_1' = y_2' = 0$. Hence, we must solve the system

$$\begin{cases} y_2^2 - 1 = 0, \\ y_2 - y_1 = 0. \end{cases}$$

From the first equation, we obtain $y_2^2 = 1$, so $y_2 = 1$ or $y_2 = -1$. From the second equation, we get $y_1 = y_2$. Therefore, if $y_2 = 1$, then $y_1 = 1$, and if $y_2 = -1$, then $y_1 = -1$. Thus, the stationary solutions are

$$\vec{y}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \vec{y}_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence, the system has exactly two stationary solutions, namely $(1, 1)$ and $(-1, -1)$ as in Figure 122.

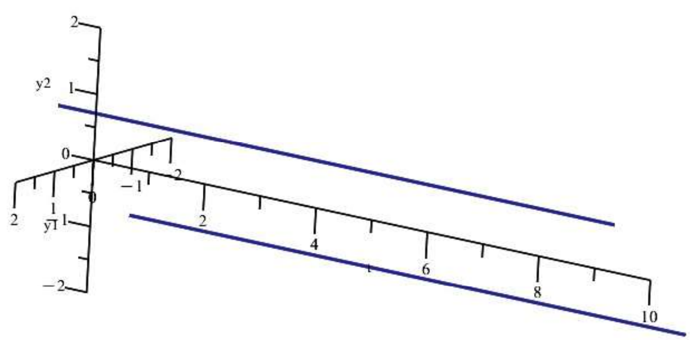

Figure 122: Stationary solutions.

We wonder now if these stationary solutions are stable or not, as we have done before. How to recognize stable solutions?

Definition 29.4. Assume that the constant function $\vec{y}_s(t) = \vec{y}_0$ is a stationary solution of the system $\vec{y}\,' = F(t, \vec{y})$. That is, the point $\vec{y}_0$ is its equilibrium. We say that this stationary solution is (asymptotically) **stable** if there is $t_s \in \mathbb{R}$ and $\delta > 0$ so that every solution $\vec{y}(t)$ such that

$$\|\vec{y}(T) - \vec{y}_s(T)\| < \delta$$

for some $T \geqslant t_s$ must necessarily exist on (T, ∞) and converges to $\vec{y}_0$ as $t \to \infty$. We say that $\vec{y}_s$ is **unstable** if it is not stable.

This definition formalizes the idea that an equilibrium is stable when every solution that starts sufficiently close to it, at least from some time onward, remains well defined for all future times and in fact approaches the equilibrium as time goes to infinity. In other words, a small perturbation of the system does not drive the solution away, but instead the motion is eventually attracted back toward the stationary state.

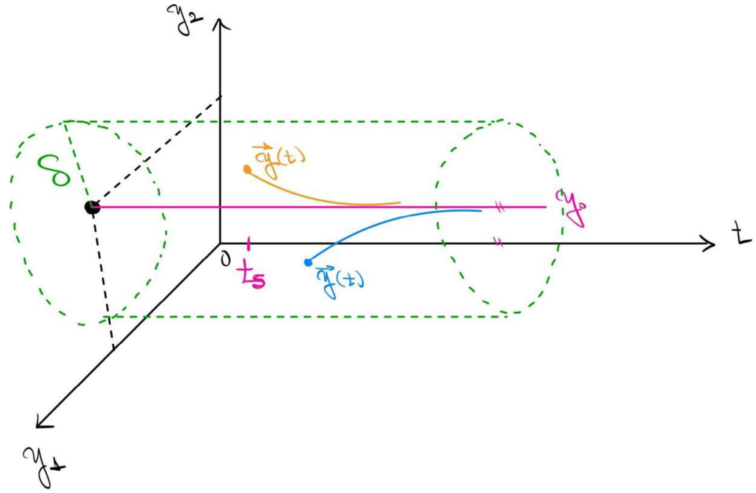

Figure 123: Geometric interpretation in 3D about stable solutions.

The presence of the parameter δ expresses the notion of being initially close to the equilibrium, while the condition $t \to \infty$ emphasizes that stability is a long-term property (see Figure 123). Thus, an asymptotically stable equilibrium is not only resistant to small disturbances, but also attractive. By contrast, if this behavior fails, then the equilibrium is called unstable, meaning that arbitrarily small perturbations may produce solutions that do not return to the stationary state.

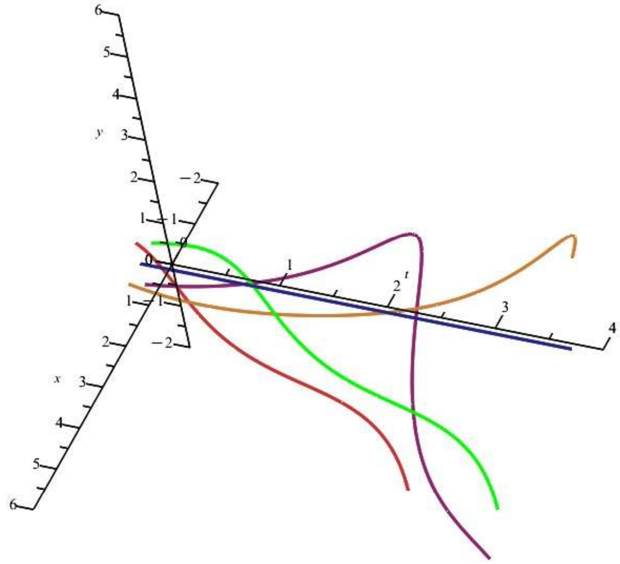

Figure 124: The stationary solutions $(1, 1)$ and $(-1, -1)$ seem to be unstable.

In Figure 124, one can see that the stationary solutions we have found in the previous example seem to be unstable.

A natural question now arises: even if the solutions we have considered so far happen to converge to a stationary solution, how can we conclude that **all** solutions starting sufficiently close to it behave in the same way? From a few particular examples alone, this cannot be guaranteed. In order to determine whether a stationary solution is stable, we need more powerful tools that allow us to analyze the behavior of entire families of solutions, and not just a small number of specific ones.

Let us consider a linear homogeneous system

$$\vec{y}' = A\vec{y}.$$

Can we find stationary solutions for this system? To answer this, we ask when $A\vec{y} = 0$. From this, we immediately see that $\vec{y}_0 = 0$ is always a stationary solution. Of course, there may exist other stationary solutions different from 0, but for the moment we shall focus only on this one. Is it stable?

We claim that $\vec{y}_0 = 0$ is stable if and only if $\vec{y}(t) \to 0$ as $t \to \infty$ for every solution $\vec{y}$. The implication "$\Leftarrow$" is immediate. For the converse, assume that $\vec{y}_0 = 0$ is stable. Then, by definition, every solution starting sufficiently close to 0 converges to 0 as $t \to \infty$. Now let $\vec{y}$ be any solution, not necessarily close to 0. Since the system is homogeneous, for any nonzero constant $a \in \mathbb{R}$, the function $\vec{y}/a$ is also a solution. Choosing a sufficiently large, the solution $\vec{y}/a$ becomes arbitrarily close to 0. By stability, it follows that

$$\frac{\vec{y}(t)}{a} \to 0 \qquad \text{as} \qquad t \to \infty.$$

Multiplying by a, we conclude that $\vec{y}(t) \to 0$ as $t \to \infty$. Hence, every solution converges to the origin as we can see in Figure 125.

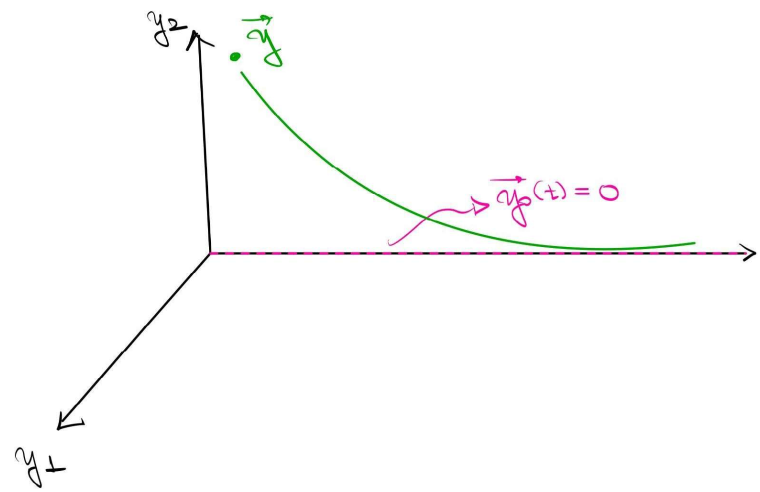

Figure 125: Behavior for a homogeneous system when $\vec{y}_0$ is stationary solution.

We can also write the statement that $\vec{y}_0 = 0$ is stable if and only if $\vec{y} \to 0$ as $t \to \infty$ for every solution $\vec{y}$. This is equivalent to saying that $\|\vec{y}\| \to 0$ as $t \to \infty$ for every solution $\vec{y}$.

In our case, once we solve the homogeneous system, we obtain a fundamental system $\{\vec{y}_1, \dots, \vec{y}_n\}$. Therefore, the general solution can be written as

$$\vec{y} = a_1\vec{y}_1 + \cdots + a_n\vec{y}_n. \tag{32}$$

In order for $\vec{y}_0 = 0$ to be stable, all solutions must converge to 0, which means that every element of the fundamental system must also converge to 0. Conversely, if every element of the fundamental system converges to 0, then the general solution (32), being a linear combination of them, also converges to 0. Therefore, everything boils down to checking whether the elements of the fundamental system converge to 0 as $t \to \infty$.

The elements of the fundamental system are of the form

$$\vec{y}_i = \vec{v}_i e^{\lambda_i t},$$

where λ_i is an eigenvalue and $\vec{v}_i$ is an eigenvector associated with λ_i. Therefore,

$$\|\vec{y}_i\| = \|\vec{v}_i e^{\lambda_i t}\| = \|\vec{v}_i\| \cdot e^{\mathrm{Re}(\lambda_i)t}.$$

This means that $\|\vec{y}_i\| \to 0$ as $t \to \infty$ if and only if $e^{\mathrm{Re}(\lambda_i)t} \to 0$ as $t \to \infty$, which happens if and only if $\mathrm{Re}(\lambda_i) < 0$. Otherwise, $\vec{y}_i$ does not converge to 0.

In fact, this also works when λ_i is an eigenvalue of higher multiplicity, since in that case the corresponding solutions are of the form

$$\vec{v}_i\, t^m e^{\lambda_i t},$$

and then

$$\|\vec{v}_i\, t^m e^{\lambda_i t}\| = \|\vec{v}_i\| \cdot |t|^m \cdot e^{\mathrm{Re}(\lambda_i)t}.$$

The dominant term is again the exponential one. Therefore, the sign of $\mathrm{Re}(\lambda_i)$ is what determines whether the solution converges to 0 or not.

Formally, we have the following result.

Theorem 29.5. Consider some homogeneous system of linear differential equations $\vec{y}\,' = A\vec{y}$ with constant coefficients. If all eigenvalues of A have negative real parts, then the equilibrium $\vec{y}_0 = \vec{0}$ is asymptotically stable. Otherwise it is unstable.

Example 29.6. Let us analyze the stability of some of the examples studied previously. For instance, in one of them we found the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$. Since both eigenvalues have positive real part, the stationary solution $\vec{y}_0 = \vec{0}$ is unstable. This is illustrated in Figure 126, where one can see that the solutions move away from the origin instead of approaching it.

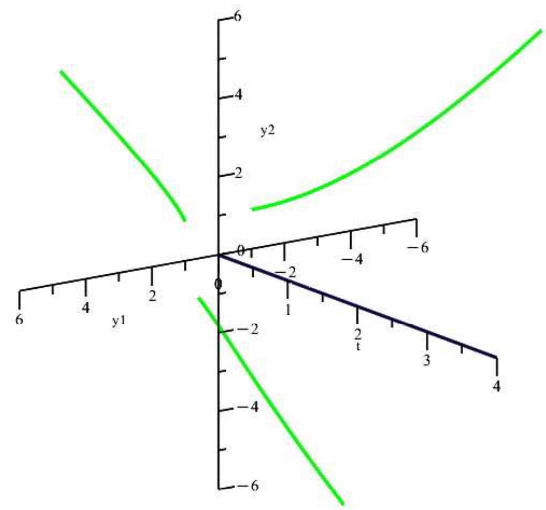

Figure 126: Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$.

We also studied an example in which the eigenvalues were complex. Notice that in Figure 127 the trajectories exhibit an oscillatory motion around the stationary solution, which is the typical geometric effect produced by complex eigenvalues. Thus, the presence of complex eigenvalues does not by itself prevent stability; what matters is the sign of their real part.

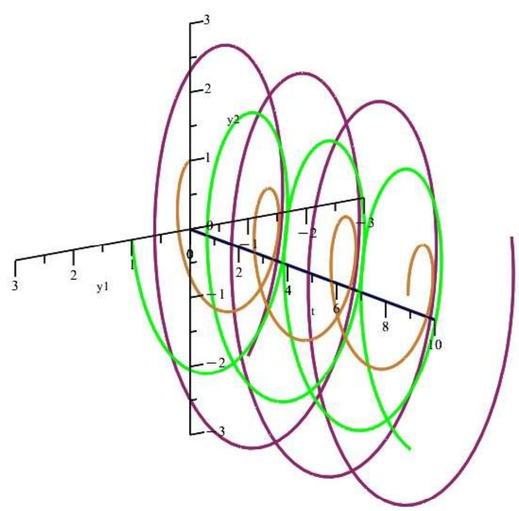

Figure 127: Complex eigenvalues.

Finally, we also considered an example in which the eigenvalue $\lambda = -2$ had higher multiplicity. Since its real part is negative, the stationary solution is again asymptotically stable, as illustrated in Figure 128.

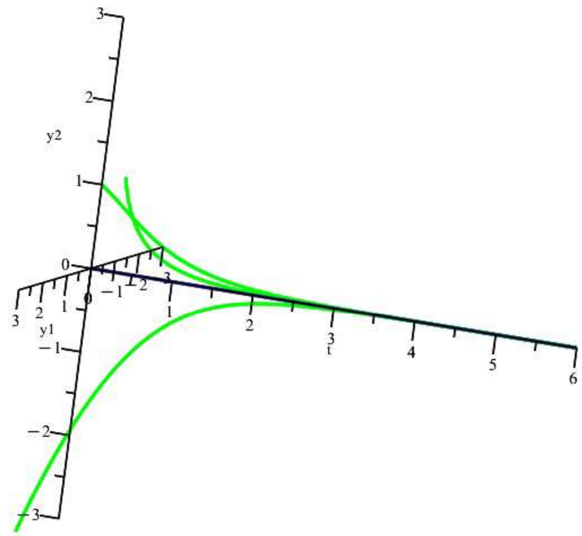

Figure 128: Higher multiplicity eigenvalue.

29.2 Solving systems of 1st order ODEs numerically

In this short section, we will solve one system of ODEs numerically using the Euler method. To do so, as we have seen before, we will approximate its solutions at discrete time steps, rather than finding an explicit formula. This is especially important in applications, since most systems arising in practice cannot be solved analytically. Starting from an initial condition, the numerical methods we have seen such as Euler's method or Runge-Kutta's, generate successive approximations of the solution and allow us to study its behavior, visualize trajectories and estimate long-term dynamics. This section should be faced as a revision of the Euler method.

Let us consider once again the system (30) of first-order ordinary differential equations. For the reader's convenience, we rewrite it here:

$$\begin{cases} y_1' = F_1(t, y_1, y_2, \ldots, y_n), \\ y_2' = F_2(t, y_1, y_2, \ldots, y_n), \\ \quad\vdots \\ y_n' = F_n(t, y_1, y_2, \ldots, y_n). \end{cases}$$

We now supplement this system with the initial conditions $y_1(t_0) = y_{1,0}, y_2(t_0) = y_{2,0}, \ldots, y_n(t_0) = y_{n,0}$. In this way, we obtain a Cauchy problem consisting of the system above together with its initial data, and our goal will be to solve it numerically. Notice that each unknown is a function of the independent variable t, that is, $y = y(t)$. Before turning to specific examples, we first introduce the general framework needed to study this problem.

Let us consider the interval $[t_0, t_0 + T]$ for some $T > 0$. Let $m \in \mathbb{N}$ be the number of subintervals, and define the step size by $h = T/m$. This yields the partition $\{t_0, t_1, \ldots, t_m\}$. The Euler method extends naturally to systems of differential equations: it is applied componentwise to each unknown function, and this is precisely the approach we will use. In this way, for each $j = 1, \ldots, n$, we obtain a sequence of approximate values

$$\{y_{j,0}, y_{j,1}, \ldots, y_{j,m}\}.$$

These approximations are computed recursively by

$$y_{j,i+1} = y_{j,i} + h\, F_j(t_i, y_{1,i}, y_{2,i}, \ldots, y_{n,i}),$$

for every $i = 0, \ldots, m-1$ and $j = 1, \ldots, n$ (see Figure 129). To illustrate this, we will apply in the next problem.

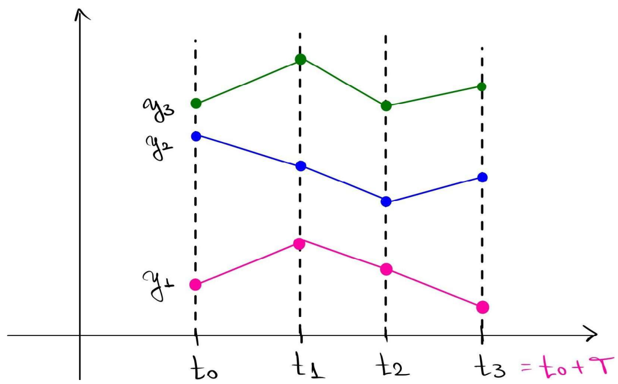

Figure 129: Numerical approach for a system of first order ODEs.

Example 29.7. We will solve numerically the following system. For Euler's method on $[0, 4]$ with partition size $m = 4$, the step size is

$$h = \frac{4-0}{4} = 1.$$

We consider the following system of ODEs

$$\begin{cases} \dot{x} = f_1(t, x, y) = y^2 - 1, \\ \dot{y} = f_2(t, x, y) = y - x. \end{cases}$$

For every j, Euler's method gives

$$x_{j+1} = x_j + h \cdot (y_j^2 - 1) \quad \text{and} \quad y_{j+1} = y_j + h \cdot (y_j - x_j).$$

Since $h = 1$, this becomes

$$x_{j+1} = x_j + (y_j^2 - 1) \quad \text{and} \quad y_{j+1} = y_j + (y_j - x_j).$$

The initial values are $t_0 = 0$, $x_0 = \frac{1}{2}$ and $y_0 = 1$. Now we compute step by step.

⋆ At $t_0 = 0$:

$$x_1 = x_0 + (y_0^2 - 1) = \frac{1}{2} + (1 - 1) = \frac{1}{2},$$

$$y_1 = y_0 + (y_0 - x_0) = 1 + \left(1 - \frac{1}{2}\right) = \frac{3}{2}.$$

⋆ At $t_1 = 1$:

$$x_2 = x_1 + (y_1^2 - 1) = \frac{1}{2} + \left(\frac{9}{4} - 1\right) = \frac{1}{2} + \frac{5}{4} = \frac{7}{4},$$

$$y_2 = y_1 + (y_1 - x_1) = \frac{3}{2} + \left(\frac{3}{2} - \frac{1}{2}\right) = \frac{5}{2}.$$

⋆ At $t_2 = 2$:

$$x_3 = x_2 + (y_2^2 - 1) = \frac{7}{4} + \left(\frac{25}{4} - 1\right) = \frac{7}{4} + \frac{21}{4} = 7,$$

$$y_3 = y_2 + (y_2 - x_2) = \frac{5}{2} + \left(\frac{5}{2} - \frac{7}{4}\right) = \frac{5}{2} + \frac{3}{4} = \frac{13}{4}.$$

⋆ At $t_3 = 3$:

$$x_4 = x_3 + (y_3^2 - 1) = 7 + \left(\frac{169}{16} - 1\right) = 7 + \frac{153}{16} = \frac{265}{16},$$

$$y_4 = y_3 + (y_3 - x_3) = \frac{13}{4} + \left(\frac{13}{4} - 7\right) = \frac{13}{4} - \frac{15}{4} = -\frac{1}{2}.$$

Therefore, the Euler approximations are

t_n	x_n	y_n
0	1/2	1
1	1/2	3/2
2	7/4	5/2
3	7	13/4
4	265/16	−1/2

We can see the approximations in Figure 130.

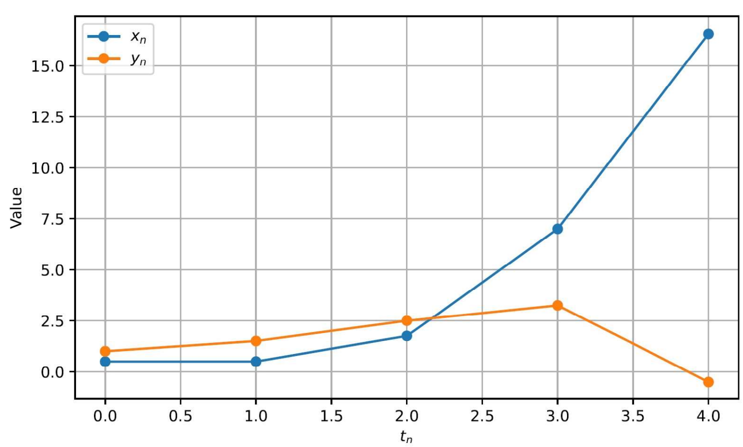

Figure 130: The Euler method in a system of ODEs.

The Euler approximation itself is displayed in Figure 131 without the exact solution for comparison. The graph shows only the polygonal numerical trajectories obtained from the values computed before.

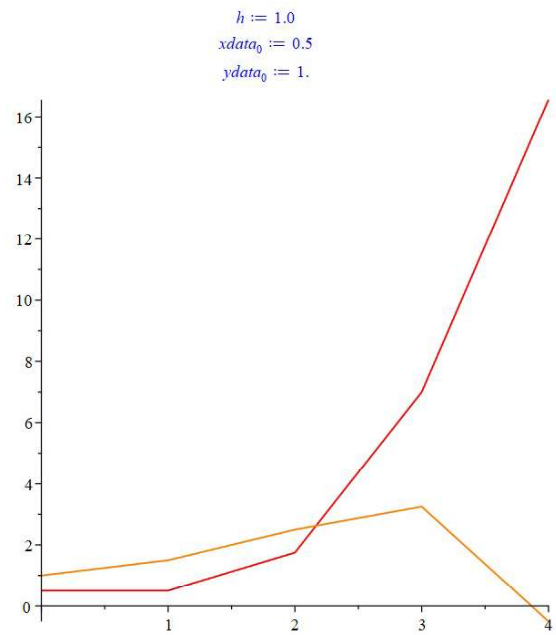

Figure 131: The Euler method in a system of ODEs on Maple.

In Figure 132, where the step size is again $h = 1$, we recover essentially the same situation as in the Euler computation we performed before. The red curve should be compared to the dark blue while the orange curve should be compared to the light blue.

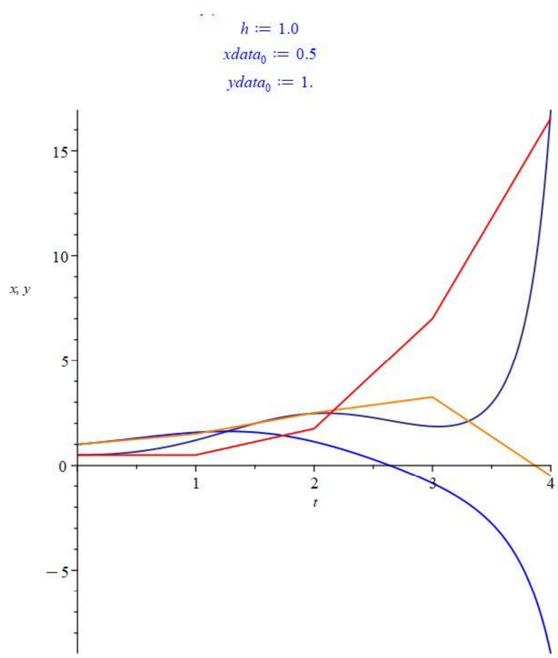

Figure 132: The Euler method in a system of ODEs on Maple with step size $h = 1$.

Finally in Figure 133, where the step size is $h = 0.2$, the Euler approximation follows the exact solution much more closely than in the computation we carried out before with $h = 1$. The qualitative behavior is already well captured: one component increases rapidly for larger values of t, while the other eventually decreases and becomes negative.

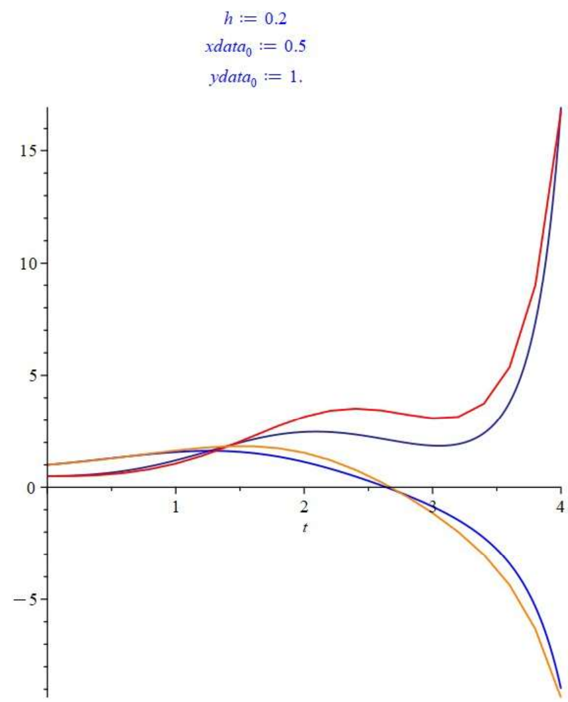

Figure 133: The Euler method in a system of ODEs on Maple with step size $h = 0.1$.