21 Practice #7: Wednesday, April 1st, 2026

21.1 Finding roots/solving equations numerically (classical methods)

21.2 Problem 1

Let us consider the function

$$f(x) = \frac{x}{x^2 + 1},$$

and suppose we want to approximate a root of this function. In fact, the only root is $x = 0$, but let us pretend that this is not known in advance. We will apply the bisection method on the interval $[-5, 3]$ and describe the first two iterations.

Starting from the initial interval $[-5, 3]$, we set

$$a_0 = -5 \qquad \text{and} \qquad b_0 = 3.$$

We first check the signs of the function at the endpoints:

$$f(-5) = -\frac{5}{26} < 0 \qquad \text{and} \qquad f(3) = \frac{3}{10} > 0.$$

Since $f(a_0)$ and $f(b_0)$ have opposite signs, the Intermediate Value Theorem guarantees that there exists at least one root in the interval $[-5, 3]$. The midpoint of this interval is

$$m_0 = \frac{a_0 + b_0}{2} = \frac{-5 + 3}{2} = -1.$$

Evaluating the function at this point, we obtain

$$f(-1) = \frac{-1}{2} < 0.$$

Since $f(-1)$ has the same sign as $f(-5)$ and the opposite sign of $f(3)$, the root must lie in the interval

$$[a_1, b_1] = [-1, 3].$$

We now repeat the procedure. The midpoint of the new interval is

$$m_1 = \frac{a_1 + b_1}{2} = \frac{-1 + 3}{2} = 1.$$

Then

$$f(1) = \frac{1}{2} > 0.$$

Since $f(1)$ has the same sign as $f(3)$ and the opposite sign of $f(-1)$, the root must lie in the interval

$$[a_2, b_2] = [-1, 1].$$

Thus, after two iterations of the bisection method, we obtain the smaller interval $[-1, 1]$, which still contains the root.

21.3 Problem 2

Consider once again the function

$$f(x) = \frac{x}{x^2+1}.$$

We now apply Newton's method with initial guess $x_0 = \frac{1}{2}$, and we will display the first three iterations. Recall that Newton's method is given by formula (23). We begin by computing the derivative of f. Using the quotient rule, we obtain

$$f'(x) = \frac{(x^2+1)\cdot 1 - x \cdot 2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}.$$

We now compute the first Newton iterate:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{1}{2} - \frac{\frac{1/2}{1/4+1}}{\frac{1-1/4}{(1/4+1)^2}} = \frac{1}{2} - \frac{\frac{1/2}{5/4}}{\frac{3/4}{25/16}} = \frac{1}{2} - \frac{2/5}{12/25} = \frac{1}{2} - \frac{5}{6} = -\frac{1}{3}.$$

Let us now simplify the Newton iteration formula in this particular case. We have

$$x_{k+1} = x_k - \frac{\frac{x_k}{x_k^2+1}}{\frac{1-x_k^2}{(x_k^2+1)^2}} = x_k - \frac{x_k(x_k^2+1)}{1-x_k^2}.$$

Putting everything over a common denominator,

$$x_{k+1} = \frac{x_k(1-x_k^2) - x_k(x_k^2+1)}{1-x_k^2} = \frac{-2x_k^3}{1-x_k^2}.$$

This form is much more convenient for the next computations. For the second iteration, we substitute $x_1 = -\frac{1}{3}$ into the simplified formula:

$$x_2 = \frac{-2\left(-\frac{1}{3}\right)^3}{1-\left(-\frac{1}{3}\right)^2} = \frac{-2\left(-\frac{1}{27}\right)}{1-\frac{1}{9}} = \frac{2/27}{8/9} = \frac{2}{27}\cdot\frac{9}{8} = \frac{1}{12}.$$

Finally, for the third iteration, we use $x_2 = \frac{1}{12}$:

$$x_3 = \frac{-2\left(\frac{1}{12}\right)^3}{1-\left(\frac{1}{12}\right)^2} = \frac{-2\cdot\frac{1}{1728}}{1-\frac{1}{144}} = \frac{-1/864}{143/144} = -\frac{1}{864}\cdot\frac{144}{143} = -\frac{1}{858}.$$

Therefore, the first three Newton iterates are

$$x_1 = -\frac{1}{3}, \quad x_2 = \frac{1}{12} \quad \text{and} \quad x_3 = -\frac{1}{858}.$$

We see that the iterates approach the root $x = 0$ very rapidly.

21.4 Problem 3

Recall that the order of convergence of an iterative method is defined through an inequality of the form

$$|E_{k+1}| \leqslant C\,|E_k|^p$$

where $E_k = r - x_k$ denotes the error at the k-th iteration, and r is the exact solution. This notion of order is completely different from the one we studied before for approximation formulas. Indeed, in that setting, the order was defined by an estimate of the form

$$|E_h| \leqslant c\,|h|^p$$

where h is is the step size, which is chosen in advance and tends to zero. The difference comes from the nature of the approximation process. Indeed,

- ⋆ In numerical differentiation and integrations, the error is measured in terms of the parameter h. There, the question is: how fast does the error go to zero as the grid is refined, that is, as $h \to 0$? Thus, the order describes the dependence of the error on the parameter h.
- ⋆ By contrast, in an iterative method such as Newton's method, there is no parameter h. The approximation is produced step by step, generating a sequence $\{x_k\}$. For this reason, the relevant question is no longer how the error depends on some external quantity, but rather how the error at one iteration depends on the error at the previous one. Here, the order p measures the speed of convergence of the iteration: if $p = 1$, the convergence is linear; if $p = 2$, it is quadratic; if $p = 3$, it is cubic; and so on.

Recall that the order of the bisection method is 1 while the order of the Newton method is 2.

21.5 Problem 4

Let us consider once again the function

$$f(x) = \frac{x}{x^2 + 1}.$$

As we saw in Problem 21.3, the second Newton iterate is

$$x_2 = \frac{1}{12}.$$

Now suppose that we require an approximation $\hat{r}$ whose error is at most $\varepsilon = \frac{1}{4}$. We may then ask whether the approximation $x_2 = \frac{1}{12}$ is already accurate enough. To answer this, we compare the values of the function at x_2 and at nearby points. First, we compute

$$f\left(\frac{1}{12}\right) = \frac{\frac{1}{12}}{\frac{1}{144} + 1} = \frac{1/12}{145/144} = \frac{12}{145} > 0.$$

Next, we look to the right of x_2 by a distance ε and find that

$$f(x_2 + \varepsilon) = f\left(\frac{1}{12} + \frac{1}{4}\right) = f\left(\frac{1}{3}\right) = \frac{\frac{1}{3}}{\frac{1}{9} + 1} = \frac{1/3}{10/9} = \frac{3}{10} > 0.$$

Since both $f(x_2)$ and $f(x_2+\varepsilon)$ are positive, there is no change of sign in the interval $[x_2, x_2+\varepsilon]$. Therefore, this information does not allow us to conclude that a root lies in that interval. Let us now move to the left instead. We obtain

$$f(x_2 - \varepsilon) = f\left(\frac{1}{12} - \frac{1}{4}\right) = f\left(-\frac{1}{6}\right) = \frac{-\frac{1}{6}}{\frac{1}{36} + 1} = \frac{-1/6}{37/36} = -\frac{6}{37} < 0.$$

Thus, $f(x_2 - \varepsilon) < 0$, while $f(x_2) > 0$. Since the function is continuous, the Intermediate Value Theorem implies that there exists a root r in the interval $[x_2 - \varepsilon, x_2]$. Hence, $|r - x_2| < \varepsilon$. We conclude that the approximation $x_2 = \frac{1}{12}$ already satisfies the required error bound, and therefore it is good enough for the prescribed tolerance.

Remark 21.1. One should note, however, that this procedure is not always reliable. Indeed, it may happen that the three values $f(x_k - \varepsilon), f(x_k)$ and $f(x_k + \varepsilon)$ all have the same sign. In that case, there is no sign change in either of the intervals $[x_k - \varepsilon, x_k]$ or $[x_k, x_k + \varepsilon]$, and therefore the Intermediate Value Theorem gives us no information about the presence of a root near x_k. This does *not* mean that there is no root within distance ε of x_k; it only means that this test is inconclusive.

We could call this test as the **three-point test**.

21.6 Problem 5

We consider the equation

$$x - 3 = \frac{1}{x^2},$$

and we would like to approximate one of its solutions numerically. We follow the requested steps.

(a) We will rewrite the equation as a root-finding problem. We begin by rewriting the equation in the form $f(x) = 0$. Moving all terms to the same side, we define $f(x) = x - 3 - \frac{1}{x^2}$.

(b) We will apply the bisection method on $[1, 5]$ and show two iterations. We start with the interval $[1, 5]$, so we set $a_0 = 1$ and $b_0 = 5$. We first evaluate the function at the endpoints:

$$f(1) = 1 - 3 - \frac{1}{1^2} = -3 < 0 \quad \text{and} \quad f(5) = 5 - 3 - \frac{1}{25} = 2 - \frac{1}{25} = \frac{49}{25} > 0.$$

Since $f(a_0)$ and $f(b_0)$ have opposite signs, the Intermediate Value Theorem guarantees that there is at least one root in the interval $[1, 5]$. The midpoint of this interval is $m_0 = 3$. Now we compute $f(3) = -\frac{1}{9} < 0$. Since $f(3)$ has the same sign as $f(1)$ and the opposite sign of $f(5)$, the root must lie in the interval $[a_1, b_1] = [3, 5]$. We now perform the second iteration. The midpoint of the new interval is $m_1 = 4$. Evaluating the function at this point, we get $f(4) = \frac{15}{16} > 0$. Since $f(4)$ has the same sign as $f(5)$ and the opposite sign of $f(3)$, the root must lie in the interval $[a_2, b_2] = [3, 4]$. Thus, after two iterations of the bisection method, we conclude that the root lies in the interval $[3, 4]$.

(c) We will apply Newton's method with $x_0 = 1$ and showing two iterations. We apply Newton's method to the function $f(x) = x - 3 - \frac{1}{x^2}$. We first compute the derivative which is given by $f'(x) = 1 + \frac{2}{x^3}$. We now compute the first iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1 - 3 - \frac{1}{1^2}}{1 + \frac{2}{1^3}} = 1 - \frac{-3}{3} = 2.$$

Next, we compute the second iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{2 - 3 - \frac{1}{2^2}}{1 + \frac{2}{2^3}} = 3.$$

(d) Suppose now that the tolerance in Newton's method is $\varepsilon = 0.5$, and we would like to determine whether the approximation found before, namely $x_2 = 3$, is already accurate enough. To do so, we apply the three-point test. We first compute

$$f(3) = 3 - 3 - \frac{1}{3^2} = -\frac{1}{9} < 0.$$

Next, we evaluate the function at the point $x_2 + \varepsilon = 3 + \frac{1}{2} = \frac{7}{2}$:

$$f(3 + \varepsilon) = f\left(\frac{7}{2}\right) = \frac{7}{2} - 3 - \frac{1}{\left(\frac{7}{2}\right)^2} = \frac{1}{2} - \frac{4}{49} = \frac{41}{98} > 0.$$

Thus, $f(3) < 0$ while $f(3+\varepsilon) > 0$. Since the function is continuous, the Intermediate Value Theorem implies that there is a root in the interval $[3, 3+\varepsilon] = [3, 3.5]$. Therefore, if r denotes the root we are looking for, then $|r - 3| < 0.5$. This shows that the approximation $x_2 = 3$ is already good enough for the prescribed tolerance.