

## 24 Practice #8: Wednesday, April 8th, 2026

### 24.1 Fixed point iteration

### 24.2 Problem 1

Let us suppose that we want to find a root of the equation

$$x^2 = \frac{1}{x+1}. \quad (27)$$

We will proceed as follows:

- transform this equation into a fixed-point problem in several different ways;
- use the corresponding iteration formulas with  $x_0 = 0$  and compute two iterations.

Let us first observe that if  $x \in \mathbb{R}$  is a solution, then, since  $x^2 \geq 0$ , we must also have

$$\frac{1}{x+1} \geq 0.$$

This implies that  $x+1 > 0$  and therefore  $x > -1$ . This is our first restriction on the possible values of  $x$ . On the other hand, notice that at  $x = 0$  the left-hand side of the equation is 0, while the right-hand side is 1. At  $x = 1$ , the left-hand side is 1, while the right-hand side is  $\frac{1}{2}$ . Therefore, the quantity

$$x^2 - \frac{1}{x+1}$$

changes sign between  $x = 0$  and  $x = 1$ , which means that there must be a root in the interval  $(0, 1)$ .

Since equation (27) can be approached in different ways, we will consider several possible formulations.

★ **First approach:** let us use the standard idea that, if  $f(x) = 0$ , then we may add  $x$  to both sides and obtain  $f(x) + x = x$ . In our case,  $f(x) = x^2 - \frac{1}{1+x}$ , and therefore

$$f(x) + x = x$$

becomes

$$x^2 - \frac{1}{1+x} + x = x.$$

We then define  $\varphi(x) = x^2 - \frac{1}{1+x} + x$ . The iteration is given by the fixed-point formula  $x_{k+1} = \varphi(x_k)$ , that is,

$$x_{k+1} = x_k^2 - \frac{1}{x_k+1} + x_k.$$

If we start with  $x_0 = 0$ , then we obtain  $x_1 = -1$ . But this immediately causes a problem in the next step, since

$$x_2 = (-1)^2 - \frac{1}{(-1)+1} + (-1) = 1 - \frac{1}{0} - 1,$$

which is not defined. This is a very bad sign, as it shows that this fixed-point formulation is not suitable for the iteration.

Let us now try starting with  $x_0 = 1$ . In this case, we obtain  $x_1 = \frac{3}{2}$  and  $x_2 = \frac{67}{20}$ , which suggests that the sequence may be diverging to  $+\infty$ . This is in fact consistent with the behavior of the derivative, recalling that for fixed-point iterations we prefer maps that are as flat as possible. Since

$$\varphi'(x) = 2x + \frac{1}{(1+x)^2} + 1,$$

we have, for instance, that  $|\varphi'(0)| = 2 > 1$ . Moreover, if  $x > 0$ , then

$$|\varphi'(x)| = 2x + \frac{1}{(1+x)^2} + 1 > 1.$$

This shows that the fixed-point map is not contractive on the region of interest. Therefore, the iteration is very likely to diverge, which agrees with the behavior suggested by the first computed terms.

★ **Second approach:** Let us now consider

$$x^2 = \frac{1}{1+x} \Leftrightarrow x = \frac{1}{\sqrt{1+x}}.$$

In this case, we define  $\varphi(x) = \frac{1}{\sqrt{1+x}}$ . The corresponding iteration is then

$$x_{k+1} = \frac{1}{\sqrt{1+x_k}}.$$

Starting with  $x_0 = 0$ , we obtain  $x_1 = 1$  and  $x_2 = \frac{1}{\sqrt{2}}$ , which already looks more promising. Indeed, notice that

$$\varphi'(x) = -\frac{1}{2} \cdot \frac{1}{(1+x)^{3/2}}.$$

Hence,  $|\varphi'(0)| = \frac{1}{2} < 1$ . In fact, for every  $x \in (0, 1)$ , we have

$$|\varphi'(x)| \leq \frac{1}{2} < 1.$$

Moreover, in this case  $\varphi : (0, 1) \rightarrow (0, 1)$ . Therefore,  $\varphi$  is a contraction on  $(0, 1)$  and the iteration converges.

★ **Third approach:** Let us now consider

$$x^2 = \frac{1}{1+x} \Leftrightarrow x+1 = \frac{1}{x^2} \Leftrightarrow x = \frac{1}{x^2} - 1.$$

We then define

$$\varphi(x) = \frac{1}{x^2} - 1 = \frac{1-x^2}{x^2}.$$

The corresponding iteration is therefore

$$x_{k+1} = \frac{1-x_k^2}{x_k^2}.$$

We immediately see that we cannot start with  $x_0 = 0$ , since the expression is not defined there. If we start instead with  $x_0 = 1$ , then  $x_1 = 0$  and we run into the same problem at the next iteration. If we start with  $x_0 = \frac{1}{2}$ , then we obtain  $x_1 = 3$  and  $x_2 = -\frac{8}{9}$ . This already suggests very unstable behavior. Indeed, since

$$\varphi'(x) = -\frac{2}{x^3}$$

we see that  $|\varphi'(x)| \rightarrow +\infty$  as  $x \rightarrow 0$ , and also we have that  $|\varphi'(1)| = 2 > 1$ . Therefore, this fixed-point map is not contractive near the relevant points, and the iteration is very likely to diverge.

★ **Fourth approach:** Let us now consider

$$x^2 = \frac{1}{1+x} \Leftrightarrow x = \frac{1}{x^2+x}.$$

We then define

$$\varphi(x) = \frac{1}{x^2+x}.$$

The corresponding iteration is therefore

$$x_{k+1} = \frac{1}{x_k^2+x_k}.$$

We immediately see that we cannot start with  $x_0 = 0$ , since the expression is not defined there. If we start instead with  $x_0 = 1$ , then we obtain  $x_1 = \frac{1}{2}$  and

$$x_2 = \frac{1}{\frac{1}{4} + \frac{1}{2}} = \frac{4}{3},$$

which already suggests that the sequence is oscillating and not settling down near the root. Indeed, since

$$\varphi'(x) = -\frac{2x+1}{(x^2+x)^2},$$

we have, for instance, that  $|\varphi'(1)| = \frac{3}{4} < 1$ , but

$$\left| \varphi' \left( \frac{1}{2} \right) \right| = \left| -\frac{2}{\left( \frac{1}{4} + \frac{1}{2} \right)^2} \right| = \frac{32}{9} > 1.$$

Therefore, the map is not contractive on the region of interest. This indicates that this fixed-point formulation is not stable and is very likely to produce a divergent iteration.

**Remark 24.1.** Let us note that there are several ways to reformulate equation (27), and each reformulation gives rise to a different fixed-point function whose iterative behavior may vary significantly. Notice also that, in the exam, it is enough to choose one such reformulation and work with it, determining the conditions under which the corresponding method converges or diverges according to the behavior of the derivative near the fixed point.

## 24.3 Problem 2

Since, for instance, the third approach above leads to a very poor iteration, let us try applying relaxation to it. Consider once again the function

$$\varphi(x) = \frac{1-x^2}{x^2}$$

and let us choose  $\lambda = \frac{1}{4}$ . Notice that

$$\varphi_{\frac{1}{4}}(x) = \frac{1}{4} \cdot \frac{1-x^2}{x^2} + \left(1 - \frac{1}{4}\right)x.$$

Therefore, the iteration becomes

$$x_{k+1} = \frac{1}{4} \cdot \frac{1-x_k^2}{x_k^2} + \frac{3}{4}x_k.$$

We immediately see that we cannot start with  $x_0 = 0$ . Instead, let us start with  $x_0 = 1$ . In this case, we obtain

$$x_0 = 1, \quad x_1 = \frac{3}{4} = 0.75, \quad \text{and} \quad x_2 = \frac{109}{144} \approx 0.756944$$

where the last number is slightly larger than  $x_1$ . This suggests that the iteration may be approaching some value, and therefore that it may converge. Let us examine this more carefully. We have

$$\varphi'_{\frac{1}{4}}(x) = \frac{1}{4} \cdot \left(-\frac{2}{x^3}\right) + \frac{3}{4}.$$

Notice that

$$\left|\varphi'_{\frac{1}{4}}(1)\right| = \left|-\frac{1}{2} + \frac{3}{4}\right| = \frac{1}{4} < 1.$$

However,

$$\left|\varphi'_{\frac{1}{4}}\left(\frac{1}{2}\right)\right| = \left|-4 + \frac{3}{4}\right| > 1.$$

This shows that relaxation with  $\lambda = \frac{1}{4}$  improves the behavior of the original iteration at least near  $x = 1$ , since the derivative there now has absolute value smaller than 1. However, the map is still not contractive on the whole interval of interest, as the derivative becomes too large near  $x = \frac{1}{2}$ . Therefore, although the first iterates suggest a better behavior than in the unrelaxed case, we cannot conclude global convergence from this choice of  $\lambda$ .

## 24.4 Problem 3

Let us consider once again the function  $\varphi$  defined by

$$\varphi(x) = \frac{1 - x^2}{x^2}$$

as in the previous problem and in the third approach from Problem 1. We now want to determine the optimal value  $\lambda_{\text{opt}}$  near some point  $x_0$ . Recall that

$$\varphi_\lambda(x) = \lambda \cdot \frac{1 - x^2}{x^2} + (1 - \lambda) \cdot x.$$

Our goal is to choose  $\lambda$  so that  $\varphi'_\lambda(x_0) = 0$ , that is, so that the relaxed function is as flat as possible near the point  $x_0$ . The question, however, is which point  $x_0$  should be chosen.

Let us first take  $x_0 = 1$ . Since

$$\varphi'_\lambda(x) = \lambda \cdot \left(-\frac{2}{x^3}\right) + (1 - \lambda)$$

imposing the condition  $\varphi'_\lambda(1) = 0$  gives

$$0 = \lambda \cdot (-2) + (1 - \lambda) \Rightarrow \lambda = \frac{1}{3}.$$

So, near  $x_0 = 1$ , we obtain the optimal  $\lambda$  given by

$$\lambda_{\text{opt}} = \frac{1}{3}.$$

What happens if we instead want  $\varphi'_\lambda\left(\frac{1}{2}\right) = 0$ ? In that case, we impose

$$0 = \lambda \cdot \left(-\frac{2}{(1/2)^3}\right) + (1 - \lambda) \Rightarrow \lambda = \frac{1}{17}.$$

Therefore, near  $x_0 = \frac{1}{2}$ , we obtain

$$\lambda_{\text{opt}} = \frac{1}{17}.$$

This shows that the optimal value of  $\lambda$  depends strongly on the point around which we want to flatten the relaxed function. In particular, choosing  $x_0 = 1$  leads to  $\lambda_{\text{opt}} = \frac{1}{3}$ , while choosing  $x_0 = \frac{1}{2}$  leads to the much smaller value  $\lambda_{\text{opt}} = \frac{1}{17}$ . This confirms that the effectiveness of the relaxation procedure is local in nature: the best choice of  $\lambda$  depends on where the iteration is taking place.

## 24.5 Problem 4

We encourage the reader to carry out the same analysis as in Problems 1, 2, and 3 for the equation

$$x - 3 = \frac{1}{x^2}$$

using  $x_0 = 1$  as the initial guess. As in the other example, we can have different situations.

$$\star \text{ (standard approach) } x - 3 = \frac{1}{x^2} \Leftrightarrow x - 3 - \frac{1}{x^2} = 0 \Leftrightarrow \underbrace{x - 3 - \frac{1}{x^2} + x}_{\varphi(x)} = x.$$

$$\star x - 3 = \frac{1}{x^2} \Leftrightarrow \underbrace{x^3 - 3x^2 + x - 1}_{\varphi(x)} = x.$$

$$\star x - 3 = \frac{1}{x^2} \Leftrightarrow x = \frac{1}{x(x-3)} \Leftrightarrow x = \frac{1}{\underbrace{x^2 - 3x}_{\varphi(x)}}$$

$$\star x - 3 = \frac{1}{x^2} \Leftrightarrow x = \underbrace{\frac{1}{x^2} + 3}_{\varphi(x)}$$