30 Practice #10: Wednesday, April 22nd, 2026

30.1 Solving non-homogeneous systems of linear ODEs

30.2 Problem 1

Consider the system

$$\begin{cases} y_1' = 2y_1 - y_2 + 4e^{4x}, \\ y_2' = -2y_1 + y_2 + 3. \end{cases}$$

We solve it as $\vec{y}' = A\vec{y} + \vec{b}(x)$, where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \vec{b}(x) = \begin{pmatrix} 4e^{4x} \\ 3 \end{pmatrix}.$$

First we solve the homogeneous system, then we will find a particular solution by guessing. The homogeneous system is $\vec{y}' = A\vec{y}$. To solve it, we find the eigenvalues of A as follows

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ -2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 2.$$

Expanding, we have that

$$(2-\lambda)(1-\lambda) - 2 = 2 - 3\lambda + \lambda^2 - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3).$$

So the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$. For $\lambda_1 = 0$, we need to solve $A\vec{v} = 0$ as follows

$$\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From $2v_1 - v_2 = 0$, we get $v_2 = 2v_1$. Taking $v_1 = 1$, we have that

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For $\lambda_2 = 3$, we solve the equation $(A - 3I)\vec{v} = 0$ as follows

$$(A - 3I)\vec{v} = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $-v_1 - v_2 = 0$, so $v_2 = -v_1$. Take $v_1 = 1$, then

$$\vec{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore the homogeneous solution is given by

$$\vec{y}_h(x) = c_1 e^{0x}\vec{v}^{(1)} + c_2 e^{3x}\vec{v}^{(2)} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In other words, we have that the solutions for the homogeneous equation

$$y_{1h}(x) = c_1 + c_2 e^{3x} \quad \text{and} \quad y_{2h}(x) = 2c_1 - c_2 e^{3x}$$

for every $x \in \mathbb{R}$. Now we proceed by finding a particular solution by guessing. As we have seen before, the non-homogeneous term is

$$\vec{g}(x) = \begin{pmatrix} 4e^{4x} \\ 3 \end{pmatrix}.$$

This has two different types of forcing: an exponential term involving e^{4x} and a constant term. Because of the e^{4x} forcing, we try a particular part of the form

$$\begin{pmatrix} Ae^{4x} \\ De^{4x} \end{pmatrix}.$$

Because of the constant forcing, we first think of trying a constant vector. However, a constant vector already appears in the homogeneous solution, since $\lambda = 0$ is an eigenvalue. That means a pure constant guess would overlap with the homogeneous solution and would not work. So we multiply by x and use a linear guess instead:

$$\begin{pmatrix} Bx + C \\ Ex + F \end{pmatrix}.$$

Putting both pieces together, we have a to find particular solutions of the form

$$y_{1p}(x) = Ae^{4x} + Bx + C \quad \text{and} \quad y_{2p}(x) = De^{4x} + Ex + F.$$

Then $y_{1p}'(x) = 4Ae^{4x} + B$ and $y_{2p}'(x) = 4De^{4x} + E$. Now substitute into the system. The first equation is $y_1' = 2y_1 - y_2 + 4e^{4x}$. Let then substitute to get

$$4Ae^{4x} + B = 2(Ae^{4x} + Bx + C) - (De^{4x} + Ex + F) + 4e^{4x}.$$

Once we simplify this last expression, we get

$$4Ae^{4x} + B = (2A - D + 4)e^{4x} + (2B - E)x + (2C - F).$$

Now we can match coefficients and get that $4A = 2A - D + 4$, $0 = 2B - E$ and $B = 2C - F$.

On the other hand, the second equation is $y_2' = -2y_1 + y_2 + 3$. We substitute and get

$$4De^{4x} + E = -2(Ae^{4x} + Bx + C) + (De^{4x} + Ex + F) + 3.$$

After simplifying the terms, we have that

$$4De^{4x} + E = (-2A + D)e^{4x} + (-2B + E)x + (-2C + F + 3).$$

By matching the coefficients we get that $4D = -2A + D$, $0 = -2B + E$ and $E = -2C + F + 3$.

After solving this system, we can find the values of A, B, C, D, E and F. Indeed, we have that $A = 3$, $D = -2$, $B = 1$ and $E = 2$. Notice that $F = 2C - 1$. At this point C is free, but that is not a problem: any change in C changes F in a way that only adds a homogeneous solution. To get one particular solution, choose the simplest value $C = 0$. Then, $F = -1$. So one particular solution is

$$y_{1p}(x) = 3e^{4x} + x \quad \text{and} \quad y_{2p}(x) = -2e^{4x} + 2x - 1.$$

Therefore, the full solution is $\vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x)$, that is,

$$y_1(x) = c_1 + c_2 e^{3x} + 3e^{4x} + x \quad \text{and} \quad y_2(x) = 2c_1 - c_2 e^{3x} - 2e^{4x} + 2x - 1$$

for every $x \in \mathbb{R}$.

30.3 Problem 2

We will solve now the same problem as Problem 1 but now using variation of parameters. We know already that the homogeneous solution is

$$\vec{y}_h(x) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We will be looking for a particular solution in the form

$$\vec{y}_p(x) = Y(x)\vec{u}(x) \quad \text{where} \quad \vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}.$$

For a system $\vec{y}\,' = A\vec{y} + \vec{b}(x)$, variation of parameters gives

$$\vec{u}\,'(x) = Y(x)^{-1}\vec{b}(x).$$

So we first compute $Y(x)^{-1}$. The fundamental matrix $Y(x)$ is given by

$$Y(x) = \begin{pmatrix} 1 & e^{3x} \\ 2 & -e^{3x} \end{pmatrix}.$$

Its determinant is $\det Y(x) = 1 \cdot (-e^{3x}) - 2e^{3x} = -3e^{3x}$. Hence

$$Y(x)^{-1} = \frac{1}{-3e^{3x}} \begin{pmatrix} -e^{3x} & -e^{3x} \\ -2 & 1 \end{pmatrix}$$

and simplifying we get

$$Y(x)^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3}e^{-3x} & -\frac{1}{3}e^{-3x} \end{pmatrix}.$$

Nowe we need to compute $\vec{u}'(x)$. We have that

$$\vec{u}\,'(x) = Y(x)^{-1}\vec{g}(x) = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3}e^{-3x} & -\frac{1}{3}e^{-3x} \end{pmatrix} \begin{pmatrix} 4e^{4x} \\ 3 \end{pmatrix}.$$

By multiplying the matrices, we get that

$$u_1'(x) = \frac{1}{3}(4e^{4x}) + \frac{1}{3}(3) = \frac{4}{3}e^{4x} + 1 \quad \text{and} \quad u_2'(x) = \frac{2}{3}e^{-3x}(4e^{4x}) - \frac{1}{3}e^{-3x}(3) = \frac{8}{3}e^{x} - e^{-3x}.$$

That is, we can write

$$\vec{u}\,'(x) = \begin{pmatrix} \frac{4}{3}e^{4x} + 1 \\ \frac{8}{3}e^{x} - e^{-3x} \end{pmatrix}.$$

In order to find $u(x)$, we integrate component by component and we get

$$u_1(x) = \int \left(\frac{4}{3}e^{4x} + 1\right) dx = \frac{1}{3}e^{4x} + x$$

where we ignore the constant of integration since it would only reproduce homogeneous terms. Also,

$$u_2(x) = \int \left(\frac{8}{3}e^{x} - e^{-3x}\right) dx = \frac{8}{3}e^{x} + \frac{1}{3}e^{-3x}.$$

Thus we may take

$$\vec{u}(x) = \begin{pmatrix} \frac{1}{3}e^{4x} + x \\ \frac{8}{3}e^{x} + \frac{1}{3}e^{-3x} \end{pmatrix}.$$

Now we can compute the particular solution as follows

$$\vec{y}_p(x) = Y(x)\vec{u}(x) = \begin{pmatrix} 1 & e^{3x} \\ 2 & -e^{3x} \end{pmatrix} \begin{pmatrix} \frac{1}{3}e^{4x} + x \\ \frac{8}{3}e^{x} + \frac{1}{3}e^{-3x} \end{pmatrix}.$$

For the first component, we get that

$$y_{1p}(x) = \left(\frac{1}{3}e^{4x} + x\right) + e^{3x}\left(\frac{8}{3}e^{x} + \frac{1}{3}e^{-3x}\right)$$

and therefore

$$y_{1p}(x) = \frac{1}{3}e^{4x} + x + \frac{8}{3}e^{4x} + \frac{1}{3} = 3e^{4x} + x + \frac{1}{3}.$$

For the second component, we get that

$$y_{2p}(x) = 2\left(\frac{1}{3}e^{4x} + x\right) - e^{3x}\left(\frac{8}{3}e^{x} + \frac{1}{3}e^{-3x}\right)$$

and therefore

$$y_{2p}(x) = \frac{2}{3}e^{4x} + 2x - \frac{8}{3}e^{4x} - \frac{1}{3} = -2e^{4x} + 2x - \frac{1}{3}.$$

So one particular solution produced by variation of parameters is

$$\vec{y}_p(x) = \begin{pmatrix} 3e^{4x} + x + \frac{1}{3} \\ -2e^{4x} + 2x - \frac{1}{3} \end{pmatrix}.$$

This differs from the particular solution found before only by a homogeneous constant vector, which is allowed.

As the general solution is given by $\vec{y}(x) = \vec{y}_h(x) + \vec{y}_p(x)$, we have that

$$y_1(x) = c_1 + c_2e^{3x} + 3e^{4x} + x + \frac{1}{3} \quad \text{and} \quad y_2(x) = 2c_1 - c_2e^{3x} - 2e^{4x} + 2x - \frac{1}{3}.$$

Since the constants are arbitrary, we can absorb the extra constant terms into the homogeneous part and write the solution more simply as

$$y_1(x) = c_1 + c_2e^{3x} + 3e^{4x} + x \quad \text{and} \quad y_2(x) = 2c_1 - c_2e^{3x} - 2e^{4x} + 2x - 1$$

for every $x \in \mathbb{R}$.