36 Practice #12: Wednesday, May 6th, 2026

36.1 Iteration methods

36.2 Problem 1

Consider the system

$$\begin{cases} x + 2y + 2z = 1, \\ x + y + z = 1, \\ x - y + z = 5. \end{cases}$$

The exact solution of this system is $x = 1, y = -2$ and $z = 2$. We will apply Jacobi and Gauss-Seidel iteration methods to try to approximate this solution. We first rewrite it in iterative form by isolating x from the first equation, y from the second equation, and z from the third equation. This gives

$$\begin{cases} x = 1 - 2y - 2z, \\ y = 1 - x - z, \\ z = 5 - x + y. \end{cases}$$

We use the initial approximation

$$\vec{x}_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

We now apply the Jacobi method. Thus, the iteration is

$$\begin{cases} x_{k+1} = 1 - 2y_k - 2z_k, \\ y_{k+1} = 1 - x_k - z_k, \\ z_{k+1} = 5 - x_k + y_k. \end{cases}$$

Starting from $\vec{x}_0 = (2, 0, 0)$, we obtain the following table

	$x = 1 - 2y - 2z$	$y = 1 - x - z$	$z = 5 - x + y$
$k = 0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 1$	$y_1 = -1$	$z_1 = 3$
$k = 2$	$x_2 = -3$	$y_2 = -3$	$z_2 = 3$
$k = 3$	$x_3 = 1$	$y_3 = 1$	$z_3 = 5$
$k = 4$	$x_4 = -11$	$y_4 = -5$	$z_4 = 5$

In other words, we have got the following vectors

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \quad \text{and} \quad \vec{x}_4 = \begin{pmatrix} -11 \\ -5 \\ 5 \end{pmatrix}$$

We see that the iterates do not approach the solution. On the contrary, they start to oscillate and grow, so the Jacobi method seem to be divergent for this formulation.

We now apply the Gauss–Seidel method. In this case, once a new value has been computed, it is used immediately in the following equations. Thus, the iteration is given by

$$\begin{cases} x_{k+1} = 1 - 2y_k - 2z_k, \\ y_{k+1} = 1 - x_{k+1} - z_k, \\ z_{k+1} = 5 - x_{k+1} + y_{k+1}. \end{cases}$$

Starting from

$$\vec{x}_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

we present a convenient way to display the computation in the following table

	$x = 1 - 2y - 2z$	$y = 1 - x - z$	$z = 5 - x + y$
$k = 0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 1$	$y_1 = 0$	$z_1 = 4$
$k = 2$	$x_2 = -7$	$y_2 = 4$	$z_2 = 16$
$k = 3$	$x_3 = -39$	$y_3 = 24$	$z_3 = 68$
$k = 4$	$x_4 = -183$	$y_4 = 116$	$z_4 = 304$

Thus, we obtain

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} -7 \\ 4 \\ 16 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} -39 \\ 24 \\ 68 \end{pmatrix} \quad \text{and} \quad \vec{x}_4 = \begin{pmatrix} -183 \\ 116 \\ 304 \end{pmatrix}.$$

Therefore, the Gauss-Seidel method seem not to converge for this formulation either.

36.3 Problem 2

Consider the system

$$\begin{cases} 2x + y + z = 2, \\ x + 2y + z = -1, \\ x + y + 4z = 7. \end{cases}$$

The solution of this system is the same as the previous system and it is given by $(1, -2, 2)$. We isolate x from the first equation, y from the second equation and z from the third equation. This gives

$$\begin{cases} x = 1 - 1/2\, y - 1/2\, z, \\ y = -1/2 - 1/2\, x - 1/2\, z, \\ z = 7/4 - 1/4\, x - 1/4\, y. \end{cases}$$

We use the initial approximation

$$\vec{x}_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

We first apply the Jacobi method. Thus, the iteration is

$$\begin{cases} x_{k+1} = 1 - 1/2\, y_k - 1/2\, z_k, \\ y_{k+1} = -1/2 - 1/2\, x_k - 1/2\, z_k, \\ z_{k+1} = 7/4 - 1/4\, x_k - 1/4\, y_k. \end{cases}$$

Starting from $\vec{x}_0 = (2, 0, 0)$, we obtain

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -3/2 \\ 5/4 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} 9/8 \\ -13/8 \\ 15/8 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} 7/8 \\ -2 \\ 15/8 \end{pmatrix} \quad \text{and} \quad \vec{x}_4 = \begin{pmatrix} 17/16 \\ -15/8 \\ 65/32 \end{pmatrix}.$$

as we have computed these values in the following table:

	$x = 1 - 1/2\, y - 1/2\, z$	$y = -1/2 - 1/2\, x - 1/2\, z$	$z = 7/4 - 1/4\, x - 1/4\, y$
$k = 0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 1$	$y_1 = -3/2$	$z_1 = 5/4$
$k = 2$	$x_2 = 9/8$	$y_2 = -13/8$	$z_2 = 15/8$
$k = 3$	$x_3 = 7/8$	$y_3 = -2$	$z_3 = 15/8$
$k = 4$	$x_4 = 17/16$	$y_4 = -15/8$	$z_4 = 65/32$

We now apply the Gauss–Seidel method. In this case, the new values are used as soon as they are computed, so the iteration becomes

$$\begin{cases} x_{k+1} = 1 - 1/2\, y_k - 1/2\, z_k, \\ y_{k+1} = -1/2 - 1/2\, x_{k+1} - 1/2\, z_k, \\ z_{k+1} = 7/4 - 1/4\, x_{k+1} - 1/4\, y_{k+1}. \end{cases}$$

Starting again from $\vec{x}_0 = (2, 0, 0)$, we obtain

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 7/4 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} 5/8 \\ -27/16 \\ 129/64 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} 107/128 \\ -493/256 \\ 2071/1024 \end{pmatrix} \quad \text{and} \quad \vec{x}_4 = \begin{pmatrix} 1951/2048 \\ -16278/8192 \\ 164565/81920 \end{pmatrix}.$$

Equivalently, using Maple, we can write it as decimal approximations

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1.75 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0.625 \\ -1.6875 \\ 2.015625 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0.8359375 \\ -1.92578125 \\ 2.0224609375 \end{pmatrix} \quad \text{and} \quad \vec{x}_4 = \begin{pmatrix} 0.95166015625 \\ -1.987060546875 \\ 2.00885009765625 \end{pmatrix}.$$

The corresponding table is

	$x = 1 - 1/2\,y - 1/2\,z$	$y = -1/2 - 1/2\,x - 1/2\,z$	$z = 7/4 - 1/4\,x - 1/4\,y$
$k = 0$	$x_0 = 2$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 1$	$y_1 = -1$	$z_1 = 7/4$
$k = 2$	$x_2 = 5/8$	$y_2 = -27/16$	$z_2 = 129/64$
$k = 3$	$x_3 = 107/128$	$y_3 = -493/256$	$z_3 = 2071/1024$
$k = 4$	$x_4 = 1951/2048$	$y_4 = -16278/8192$	$z_4 = 164565/81920$

In this example, both methods converge to the exact solution $(1, -2, 2)$ but the Gauss–Seidel method approaches it faster.

36.4 Problem 3

Consider the system

$$\begin{cases} x + 4y - z = 12, \\ x + 2y + 4z = -2, \\ 2x - y + z = 0. \end{cases}$$

We will

(a) reorder the system to increase the chance that iterative methods will work.

(b) derive iterative formulas.

(c) apply the Jacobi method with $\vec{x}_0 = (0, 0, 0)$ and show 2 iterations.

(d) apply the Gauss–Seidel method with $\vec{x}_0 = (0, 0, 0)$ and show 2 iterations.

Let us recall from the theory that, in the iterative method we have learned, it is convenient to arrange the equations in such a way that the diagonal coefficients are as large as possible in absolute value. The intuition behind that is that, after isolating the variables, the coefficients are divided by the diagonal entry, so larger diagonal terms usually lead to smaller iteration coefficients and therefore the possibilities of convergence increase. For this reason, we reorder the system as

$$\begin{cases} 2x - y + z = 0, \\ x + 4y - z = 12, \\ x + 2y + 4z = -2. \end{cases}$$

Isolating x from the first equation, y from the second equation, and z from the third equation, we obtain

$$\begin{cases} x = 1/2\, y - 1/2\, z, \\ y = 3 - 1/4\, x + 1/4\, z, \\ z = -1/2 - 1/4\, x - 1/2\, y. \end{cases}$$

The Jacobi iteration is therefore

$$\begin{cases} x_{k+1} = 1/2\, y_k - 1/2\, z_k, \\ y_{k+1} = 3 - 1/4\, x_k + 1/4\, z_k, \\ z_{k+1} = -1/2 - 1/4\, x_k - 1/2\, y_k. \end{cases}$$

Starting from

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we get

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 3 \\ -1/2 \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} 7/4 \\ 23/8 \\ -2 \end{pmatrix}.$$

A convenient table is

	$x = 1/2\,y - 1/2\,z$	$y = 3 - 1/4\,x + 1/4\,z$	$z = -1/2 - 1/4\,x - 1/2\,y$
$k = 0$	$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 0$	$y_1 = 3$	$z_1 = -1/2$
$k = 2$	$x_2 = 7/4$	$y_2 = 23/8$	$z_2 = -2$

For the Gauss-Seidel method, we use each new value as soon as it is computed, so the iteration becomes

$$\begin{cases} x_{k+1} = 1/2\,y_k - 1/2\,z_k, \\ y_{k+1} = 3 - 1/4\,x_{k+1} + 1/4\,z_k, \\ z_{k+1} = -1/2 - 1/4\,x_{k+1} - 1/2\,y_{k+1}. \end{cases}$$

Again starting from

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we obtain

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} 5/2 \\ 15/8 \\ -33/16 \end{pmatrix}.$$

The corresponding table is

	$x = 1/2\,y - 1/2\,z$	$y = 3 - 1/4\,x + 1/4\,z$	$z = -1/2 - 1/4\,x - 1/2\,y$
$k = 0$	$x_0 = 0$	$y_0 = 0$	$z_0 = 0$
$k = 1$	$x_1 = 0$	$y_1 = 3$	$z_1 = -2$
$k = 2$	$x_2 = 5/2$	$y_2 = 15/8$	$z_2 = -33/16$

From these first iterations, both methods appear to be convergent toward the exact solution $(2, 2, -2)$. Indeed, the Jacobi iterates

$$\begin{pmatrix} 0 \\ 3 \\ -1/2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 7/4 \\ 23/8 \\ -2 \end{pmatrix}$$

already move in the direction of the solution, while the Gauss–Seidel iterates

$$\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5/2 \\ 19/8 \\ -27/16 \end{pmatrix}$$

seem to approach it even more rapidly. This suggests that the reordered system is well suited for both iterative methods.