DEN Homework # 9

Solve the problems and then bring your work to the lab in the eleventh week of school.
Actually, it is a good idea to complete the first problem before the midterm in week 10.

1. Find the solution of the initial value problem
$$\begin{aligned} y_1' &= 5y_1 - 6y_2 \qquad & y_1(0) &= 5 \\ y_2' &= 2y_1 - 2y_2, \qquad & y_2(0) &= 3. \end{aligned}$$
Use the matrix approach.
Is the trivial stationary solution $y_1(x) = 0$, $y_2(x) = 0$ of this system stable?

2. Find a general solution of the system
$$\begin{aligned} y_1' &= 9y_2 \\ y_2' &= -y_1, \end{aligned}$$
Use the matrix approach.
Is the trivial stationary solution $y_1(x) = 0$, $y_2(x) = 0$ of this system stable?

3. Rewrite the equation $y''' + xy'' - y' - \ln(x)y = 13e^x$ as a system of linear equations of order 1.

Solution

1. $A = \begin{pmatrix} 5 & -6 \\ 2 & -2 \end{pmatrix}$, $\det(A - \lambda E) = \begin{vmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$, $\lambda = 1, 2$.

$\underline{\lambda = 1}$: $\begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$, $4v_1 - 6v_2 = 0$, choose $v_2 = 2 \implies v_1 = 3$, $\vec{y}_a = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{1\cdot x} = \begin{pmatrix} 3e^x \\ 2e^x \end{pmatrix}$.

$\underline{\lambda = 2}$: $\begin{pmatrix} 3 & -6 \\ 2 & -4 \end{pmatrix}$, $3v_1 - 6v_2 = 0$, choose $v_2 = 1 \implies v_1 = 2$, $\vec{y}_b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2x} = \begin{pmatrix} 2e^{2x} \\ e^{2x} \end{pmatrix}$.

General solution $\vec{y}(x) = a\vec{y}_a + b\vec{y}_b = \begin{pmatrix} 3a\,e^x + 2b\,e^{2x} \\ 2a\,e^x + b\,e^{2x} \end{pmatrix}$, $x \in \mathbb{R}$. Rewrite:
$y_1(x) = 3a\,e^x + 2b\,e^{2x}$, $y_2(x) = 2a\,e^x + b\,e^{2x}$, $x \in \mathbb{R}$.
Init. conditions: Equations $3a + 2b = 5$, $2a + b = 3$, then $a = b = 1$.
Solution: $y_1(x) = 3e^x + 2e^{2x}$, $y_2(x) = 2e^x + e^{2x}$, $x \in \mathbb{R}$.
Since the general solution does not satisfy $\vec{y}(x) \to \vec{0}$ as $x \to \infty$, the solution $\vec{y}(x) = \vec{0}$ is not stable.
Another good justification: Since we have an eqigenvalue $\lambda \geq 0$, it follows that ...

2. $A = \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}$, $\det(A - \lambda E) = \begin{vmatrix} 0-\lambda & 9 \\ -1 & 0-\lambda \end{vmatrix} = \lambda^2 + 9 = 0$, $\lambda = \pm 3i$.

$\underline{\lambda = 3i}$ (just one enough): $\begin{pmatrix} -3i & 9 \\ -1 & -3i \end{pmatrix}$, $-3iv_1 + 9v_2 = 0$, choose $v_2 = i \implies v_1 = 3$,

$\vec{y}_C = \begin{pmatrix} 3 \\ i \end{pmatrix} e^{3ix}$. Rewrite $e^{3ix} = \cos(3x) + i\sin(3x)$, multiply this and the vector to obtain

$\vec{y}_C(x) = \begin{pmatrix} 3\cos(3x) + 3i\sin(3x) \\ -\sin(3x) + i\cos(3x) \end{pmatrix}$. Two solutions for the basis:

$\vec{y}_a(x) = \mathrm{Re}(\vec{y}_C) = \begin{pmatrix} 3\cos(3x) \\ -\sin(3x) \end{pmatrix}$, $\qquad \vec{y}_b(x) = \mathrm{Im}(\vec{y}_C) = \begin{pmatrix} 3\sin(3x) \\ \cos(3x) \end{pmatrix}$.

Hence a general solution $\vec{y}(x) = a\vec{y}_a(x) + b\vec{y}_b(x)$, that is,
$y_1(x) = 3a\cos(3x) + 3b\sin(3x)$,
$y_2(x) = -a\sin(3x) + b\cos(3x)$, $x \in \mathbb{R}$.
Since the general solution does not satisfy $\vec{y}(x) \to \vec{0}$ as $x \to \infty$ (or, because there is an eigenvalue with $\mathrm{Re}(\lambda) \geq 0$), the solution $\vec{y}(x) = \vec{0}$ is not stable.

3. Rename: $y = y_1$. Then reduce
$$\begin{aligned} &y_1' = y_2 \\ &y_2'' + xy_2' - y_2 - \ln(x)y_1 = 13e^x \end{aligned} \implies \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &+ xy_3 - y_2 - \ln(x)y_1 = 13e^x \end{aligned}$$
Answer:
$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= \ln(x)y_1 + y_2 - xy_3 + 13e^x. \end{aligned}$$