DEN Homework # 12

Solve the problems and then bring your work to the lab in the last week of school.

1. We are given the following system:

$$\begin{aligned} x \quad\; + 2z &= -2 \\ 3x + y - z &= 9 \\ x + 4y + z &= 4. \end{aligned}$$

By the way, its solution is $x = 2$, $y = 1$, $z = -2$.

a) Rewrite this system into a form suitable for the Jacobi iterative method. Given the initial vector $x_0 = y_0 = z_0 = 0$, calculate the next three iterations of the Jacobi iteration. Do you get the feeling that it would converge to the solution?

b) Apply the Gauss-Seidel iteration to this system. Namely, calculate the first two iterations given the initial vector $x_0 = y_0 = z_0 = 0$. Add a remark outlining the key difference compared to the Jacobi iteration at a suitable place.

c) Reorder the system so that the Jacobi and Gauss-Seidel iterations have a high chance of success. Explain what shape of a system you are trying to achieve.

d) Apply the Jacobi iteration to the reordered system and calculate the next three iterations based on the initial vector $x_0 = y_0 = z_0 = 0$. Does it look hopeful?

e) Apply the Gauss-Seideliteration to the reordered system and calculate the next two iterations based on the initial vector $x_0 = y_0 = z_0 = 0$. Does it look hopeful?

Solution

1. a) The standard way to iteration is

$$\begin{aligned} x &= -2 - 2z \\ y &= 9 - 3x + z \\ z &= 4 - x - 4y. \end{aligned}$$

We calculate:

$$\begin{matrix} x_0 = 0 & & x_1 = -2 - 2z_0 = -2 & & x_2 = -2 - 2z_1 = -10 & & x_3 = 58 \\ y_0 = 0 & \Longrightarrow & y_1 = 9 - 3x_0 + z_0 = 9 & \Longrightarrow & y_2 = 9 - 3x_1 + z_1 = 19 & \Longrightarrow & y_3 = 9 \\ z_0 = 0 & & z_1 = 4 - x_0 - 4y_0 = 4 & & z_2 = 4 - x_1 - 4y_1 = -30 & & z_3 = -62. \end{matrix}$$

It does not seem likely that this would converge to the solution from part a).

b) The Gauss-Seidel iteration uses the same formulas as in part b):

$$\begin{aligned} x &= -2 - 2z \\ y &= 9 - 3x + z \\ z &= 4 - x - 4y. \end{aligned}$$

We calculate, and we always use the latest information:

$$\begin{matrix} x_0 = 0 & & x_1 = -2 - 2z_0 = -2 \\ y_0 = 0 & \Longrightarrow & y_1 = 9 - 3x_1 + z_0 = 15 \\ z_0 = 0 & & z_1 = 4 - x_1 - 4y_1 = -54. \end{matrix}$$

The difference compared to the Jacobi iteration: we used the new values immediately after finding them.

$$\begin{matrix} & x_2 = -2 - 2z_1 = 106 \\ \Longrightarrow & y_2 = 9 - 3x_2 + z_1 = -363 \\ & z_2 = 4 - x_2 - 4y_2 = 1350. \end{matrix}$$

This does not look any better.

c) When it comes to convergence of iterative methods, we appreciate systems with a strong diagonal. It is possible to rearrange the given system in this way.

$$\begin{aligned} 3x + y - z &= 9 \\ x + 4y + z &= 4 \\ x \quad\;\; + 2z &= -2. \end{aligned}$$

d) We get the formulas

$$\begin{aligned} x &= 3 - \tfrac{1}{3}y + \tfrac{1}{3}z \\ y &= 1 - \tfrac{1}{4}x - \tfrac{1}{4}z \\ z &= -1 - \tfrac{1}{2}x. \end{aligned}$$

We calculate:

$$\begin{matrix} x_0 = 0 & & x_1 = 3 - \frac{1}{3}y_0 + \frac{1}{3}z_0 = 3 & & x_2 = 3 - \frac{1}{3}y_1 + \frac{1}{3}z_1 = \frac{7}{3} & & x_3 = 2 \\ y_0 = 0 & \Longrightarrow & y_1 = 1 - \frac{1}{4}x_0 - \frac{1}{4}z_0 = 1 & \Longrightarrow & y_2 = 1 - \frac{1}{4}x_1 - \frac{1}{4}z_1 = \frac{1}{2} & \Longrightarrow & y_3 = \frac{25}{24} \\ z_0 = 0 & & z_1 = -1 - \frac{1}{2}x_0 = -1 & & z_2 = -1 - \frac{1}{2}x_1 = -\frac{5}{2} & & z_3 = -\frac{13}{6}. \end{matrix}$$

This could converge to $x = 2$, $y = 1$ and $z = -2$.

e) The Gauss-Seidel iteration uses the same formulas:

$$\begin{matrix} x_0 = 0 & & x_1 = 3 - \frac{1}{3}y_0 + \frac{1}{3}z_0 = 3 & & x_2 = \frac{25}{12} \\ y_0 = 0 & \Longrightarrow & y_1 = 1 - \frac{1}{4}x_1 - \frac{1}{4}z_0 = \frac{1}{4} & \Longrightarrow & y_2 = \frac{53}{48} \\ z_0 = 0 & & z_1 = -1 - \frac{1}{2}x_1 = -\frac{5}{2} & & z_2 = -\frac{49}{24}. \end{matrix}$$

These iterations also look like converging to $x = 2$, $y = 1$, $z = -2$.