DEN Homework # 8

Solve the problems and then bring your work to the lab in the nineth week of school.

1. In the previous homework we looked for the root of the function $f(x) = x^2 - 3x + 1$ using the bisection method and the Newton method, there we had the initial guess $x_0 = 1$.

a) Find some transformation of the equation $x^2 - 3x + 1 = 0$ into a fixed-point problem. Write the corresponding iterative formula and show two iterative steps with initial value $x_0 = 1$ (that is, find x_1, x_2).
Then make a guess regarding the success of this iteration using $|\varphi'(1)|$.

b) Repeat the task a) for some other transformation of that equation into a fixed-point problem.

Solution

1. There are quite a few possible transformations, hopefully you can find yours among those below.
• The standard approach by adding x: $x^2 - 2x + 1 = x$.
We have $\varphi(x) = x^2 - 2x + 1$, the iteration is $x_{k+1} = x_k^2 - 2x_k + 1$.
Approximations are $x_0 = 1$ (given), $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, ...
$\varphi'(x) = 2x - 2$, hence $|\varphi'(1)| = 0$. This looks hopeful, but as we see above, the iteration enters a cycle.
• $x^2 - 3x + 1 = 0$ means $3x = x^2 + 1$, hence $x = \frac{1}{3}(x^2 + 1)$.
We have $\varphi(x) = \frac{1}{3}(x^2 + 1)$, the iteration is $x_{k+1} = \frac{1}{3}(x_k^2 + 1)$.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{2}{3}$, $x_2 = \frac{13}{27}$.
(The thirteen was not planned, just a lucky coincidence.)
$\varphi'(x) = \frac{2}{3}x$, hence $|\varphi'(1)| = \frac{2}{3}$. This number is somewhat smaller than one, so things look hopeful.
By the way, this iteration converges to the root ≈ 0.38.
• $x^2 - 3x + 1 = 0$ means $x^2 = 3x - 1$, hence $x = \sqrt{3x - 1}$.
We have $\varphi(x) = \sqrt{3x - 1}$, the iteration is $x_{k+1} = \sqrt{3x_k - 1}$.
This is a bit risky, hopefully negative numbers will not appear during iteration.
Approximations are $x_0 = 1$ (given), $x_1 = \sqrt{2}$, $x_2 = \sqrt{3\sqrt{2} - 1}$.
$\varphi'(x) = \frac{3}{2\sqrt{3x-1}}$, hence $|\varphi'(1)| = \frac{3}{2\sqrt{2}} \approx 1.06$. We have a number larger than 1, this is not very hopeful. On the other hand, it is close to 1 and those x_k then move elsewhere, perhaps it will be better there.
By the way, this iteration converges to the root ≈ 2.62.
• $x^2 - 3x + 1 = 0$ means $1 = 3x - x^2$, hence $x = 3x^2 - x^3$.
We have $\varphi(x) = 3x^2 - x^3$, the iteration is $x_{k+1} = 3x_k^2 - x_k^3$.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = 4$.
$\varphi'(x) = 6x - 3x^2$, hence $|\varphi'(1)| = 3$. We have a number markedly larger than 1, this is not very hopeful.
By the way, this iteration diverges quickly.
• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x - 3) = -1$, hence for instance $x = \frac{-1}{x-3} = \frac{1}{3-x}$.
We have $\varphi(x) = \frac{1}{3-x}$, the iteration is $x_{k+1} = \frac{1}{3-x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{1}{2}$, $x_2 = \frac{2}{5}$.
$\varphi'(x) = \frac{1}{(3-x)^2}$, hence $|\varphi'(1)| = \frac{1}{4}$. This is significantly less than one, this looks quite hopeful.
By the way, this iteration converges to the root ≈ 0.38.
• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x - 3) = -1$, this means $x - 3 = -\frac{1}{x}$, hence $x = 3 - \frac{1}{x}$.
We have $\varphi(x) = 3 - \frac{1}{x}$, the iteration is $x_{k+1} = 3 - \frac{1}{x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = \frac{5}{2}$.
$\varphi'(x) = \frac{1}{x^2}$, hence $|\varphi'(1)| = 1$. This is exactly the borderline. As iterations move (hopefully) towards a root (which one?), the derivative is bound to change and we have no idea how. So no guessing here.
By the way, this iteration converges to the root ≈ 2.62.
• $x^2 - 3x + 1 = 0$ means $x^2 - 2x + 1 = x$ (we have already been here), but then $(x - 1)^2 = x$, so $x - 1 = \pm\sqrt{x}$, hence $x = 1 \pm \sqrt{x}$.
We try the one with plus: $\varphi(x) = 1 + \sqrt{x}$, the iteration is $x_{k+1} = 1 + \sqrt{x_k}$.

Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = 1 + \sqrt{2}$.
$\varphi'(x) = \frac{1}{2\sqrt{x}}$, hence $|\varphi'(1)| = \frac{1}{2}$. This is markedly smaller than one, this looks quite hopeful.
By the way, this iteration converges to the root ≈ 2.62.
We try the one with minus: $\varphi(x) = 1 - \sqrt{x}$, the iteration is $x_{k+1} = 1 - \sqrt{x_k}$.
Approximations are $x_0 = 1$ (given), $x_1 = 0$, $x_2 = 1$. We are in a cycle again.
$\varphi(x) = -\frac{1}{2\sqrt{x}}$, hence $|\varphi'(1)| = \frac{1}{2}$. This is less than one and looks hopeful, but as we saw, it did not work out at the end.
However, if we start at $x_0 = 0.9$, we reach the root ≈ 0.38.

• $x^2 - 3x + 1 = 0$ means $x^2 - x + 1 = 2x$, so $\frac{1}{2}(x^2 - x + 1) = x$.
We have $\varphi(x) = \frac{1}{2}(x^2 - x + 1)$, the iteration is $x_{k+1} = \frac{1}{2}(x_k^2 - x_k + 1)$.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{1}{2}$, $x_2 = \frac{3}{8}$.
$\varphi'(x) = \frac{1}{2}(2x - 1)$, hence $|\varphi'(1)| = \frac{1}{2}$. This is markedly smaller than one, this looks quite hopeful.
By the way, this iteration converges to the root ≈ 0.38.

• $x^2 - 3x + 1 = 0$ means $x^2 + 2x + 1 = 5x$, so $x = \frac{1}{5}(x+1)^2$.
We have $\varphi(x) = \frac{1}{5}(x+1)^2$, the iteration is $x_{k+1} = \frac{1}{5}(x_k + 1)^2$.
Approximations are $x_0 = 1$ (given), $x_1 = \frac{4}{5}$, $x_2 = \frac{81}{125}$.
$\varphi'(x) = \frac{2}{5}(x+1)$, hence $|\varphi'(1)| = \frac{4}{5}$. This is slightly less than one, this looks mildly hopeful.
By the way, this iteration converges to the root ≈ 0.38.

• $x^2 - 3x + 1 = 0$ means $x^2 - 4x + 4 = 3 - x$, that is, $(x-2)^2 = 3 - x$, so $x = 3 - (x-2)^2$.
We have $\varphi(x) = 3 - (x-2)^2$, the iteration is $x_{k+1} = 3 - (x_k - 2)^2$.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, we've got into a cycle.
$\varphi'(x) = -2(x-2)$, hence $|\varphi'(1)| = -2$. This does not look good.

• $x^2 - 3x + 1 = 0$ means $x^2 - 3x = -1$, so $x(x-3) = -1$, then $x - 3 = -\frac{1}{x}$, hence $x = 3 - \frac{1}{x}$.
We have $\varphi(x) = 3 - \frac{1}{x}$, the iteration is $x_{k+1} = 3 - \frac{1}{x_k}$.
Hopefully we will not hit a division by zero situaton.
Approximations are $x_0 = 1$ (given), $x_1 = 2$, $x_2 = \frac{5}{2}$.
$\varphi'(x) = \frac{1}{x^2}$, hence $|\varphi'(1)| = 1$. This is exactly the borderline. As iterations move (hopefully) towards a root (which one?), the derivative is bound to change and we have no idea how. So no guessing here.
By the way, this iteration converges to the root ≈ 2.62.

• $x^2 - 3x + 1 = 0$ means $x^2 - 3x + 2 = 1$, that is, $(x-2)(x-1) = 1$, hence $x - 1 = \frac{1}{x-2}$ and so $x = \frac{1}{x-2} + 1$.
We have $\varphi(x) = \frac{1}{x-2} + 1$, the iteration is $x_{k+1} = \frac{1}{x_k - 2} + 1$.
Approximations are $x_0 = 1$ (given), $x_1 = 0$, $x_2 = \frac{1}{2}$.
$\varphi'(x) = \frac{-1}{(x-2)^2}$, hence $|\varphi'(1)| = 1$. Thi is exactly the borderline, no guess gere.
By the way, this iteration converges to the root ≈ 0.38.