Let K and S be compact Hausdorff spaces. Consider $\mathcal{C}(K)$ and $\mathcal{C}(S)$ as complex Banach spaces with the supremum norm.

Goal: For every $0<\varepsilon<1$, there is $\eta(\varepsilon)>0$ such that whenever $T_0 \in S_{\mathcal{L}(\mathcal{C}(K),\mathcal{C}(S))}$ and $f_0 \in S_{\mathcal{C}(K)}$ satisfy $\|T_0(f_0)\| > 1-\eta(\varepsilon)$, there are $T \in S_{\mathcal{L}(\mathcal{C}(K),\mathcal{C}(S))}$ and $h \in S_{\mathcal{C}(K)}$ such that

$$\|T(h)\| = 1 \ , \ \|T-T_0\| < \varepsilon \text{ and } \|h-f_0\| < \varepsilon.$$

Notation: Let $M(K) := \mathcal{C}(K)^*$ be the space of complex regular Borel measures on K with the total variation norm. If $\nu \in M(K)$, then we write

$$d\nu = \theta \, d|\nu|$$

for a polar decomposition, where $|\theta| = 1$ $|\nu|$- a.e.. Recall that the value of θ on a set that has $|\nu|$-measure does not affect the measure ν, nor any integral with respect to $|\nu|$.

Every operator $T \in \mathcal{L}(\mathcal{C}(K),\mathcal{C}(S))$ has a unique representing map $\mu : S \to M(K)$ which is w^*-continuous and satisfies

$$[Tf](s) = \int_K f \, d\mu(s) \qquad (f \in \mathcal{C}(K),\ s \in S).$$

Moreover, $\|T\| = \|\mu\| := \sup_{s \in S} \|\mu(s)\|$.

$\boxed{\text{Lemmas}}$ The first lemma is basically "Lema 1.1" from the version on arxiv.

Lemma 1: Let K be a compact Hausdorff space and let $g \in \mathcal{C}(K)$ be real-valued with $g \geq 0$. Then,

$$\nu \in M(K) \longmapsto \int_K g \, d|\nu|$$

is w^*-lower semicontinuous on $M(K)$. Consequently, if $\mu : S \longrightarrow M(K)$ is w^*-continuous, then

$$s \in S \longmapsto \int_K g \, d|\mu(s)|$$

is lower semicontinuous with respect to the original topology of S.

Proof: Let $\Phi : M(K) \longrightarrow \mathbb{R}$ be defined by

$$\Phi(\nu) := \int_K g \, d|\nu|.$$

By your lemma 1.1, Φ is w^*-lower semicontinuous. We want to prove that $s \in S \mapsto \int_K g \, d|\mu(s)|$ is lower semicontinuous. This map is precisely $\Phi \circ \mu : S \to \mathbb{R}$. That is,

$$(\Phi \circ \mu)(s) = \Phi(\mu(s)) = \int_K g \, d|\mu(s)| \quad , \forall s \in S.$$

Fix $\alpha \in \mathbb{R}$. To prove lower semicontinuity, it is enough to show that the set

$$\left\{ s \in S : \int_K g \, d|\mu(s)| > \alpha \right\}$$

is open in S. Using the definition of Φ, we have

$$\left\{ s \in S : \int_K g \, d|\mu(s)| > \alpha \right\} = \{ s \in S : \Phi(\mu(s)) > \alpha \}$$

$$= \mu^{-1}\left(\{ \nu \in M(K) : \Phi(\nu) > \alpha \} \right).$$

Since Φ is w^*-lower semicontinuous, the set $\{ \nu \in M(K) : \Phi(\nu) > \alpha \}$ is w^*-open in $M(K)$. Since $\mu : S \to M(K)$ is w^*-star continuous, its preimage under μ is open in S. Hence $s \mapsto \int_K g \, d|\mu(s)|$ is lower semicontinuous.

$\blacksquare$

Lemma 2: Let K and S be compact Hausdorff spaces. Let $\mu : S \longrightarrow M(K)$ be w^*-continuous with $\|\mu\| = 1$. Let $f_0 \in S_{C(K)}$, let $s_0 \in S$ and suppose that

$$\mathrm{Re} \int_K f_0 \, d\mu(s_0) > 1 - \alpha$$

for some $\alpha > 0$. Let $0 < a < b < c < 1$, $\tau > 0$ and put

$$m := \frac{\alpha}{1-c} + \tau \quad \text{and} \quad d := (1-a) + \alpha + m.$$

Then, there exist $h \in S_{C(K)}$, a closed set $A \subseteq K$, a non-empty open set $\mathcal{U} \subseteq S$ and a w^*-continuous map $\mu_1 : S \longrightarrow M(K)$ such that

$$\{t \in K : |f_0(t)| \geq b\} \subseteq K \setminus A \subseteq \{t \in K : |f_0(t)| > a\}$$

and

(1) $\|h - f_0\| \leq 1 - a$,

(2) $|h(t)| = 1, \ \forall t \in K \setminus A$,

(3) $K \setminus A \neq \emptyset$,

(4) $|\mu_1(s)|(A) = 0, \ \forall s \in \mathcal{U}$,

(5) $\|\mu_1\| \leq 1$,

(6) $\|\mu_1 - \mu\| < m$ and

(7) $\mathrm{Re} \int_K h \, d\mu_1(s) > 1 - d, \ \forall s \in \mathcal{U}$.

<u>Proof:</u> Let us define

$$F_b := \{t \in K : |f_0(t)| \geq b\} \quad \text{and} \quad O_a := \{t \in K : |f_0(t)| > a\}.$$

Then, F_b is closed, O_a is open and $F_b \subseteq O_a$ since $a < b$. Since compact Hausdorff spaces are normal and, in particular, regular, there is an open set V such that $F_b \subseteq V \subseteq \overline{V} \subseteq O_a$. By Urysohn's lemma, applied to the disjoint closed sets F_b and $K \setminus V$, there exists a real-valued function $u \in \mathcal{C}(K, [0,1])$ such that

$$u \equiv 1 \text{ on } F_b \quad \text{and} \quad u \equiv 0 \text{ on } K \setminus V.$$

This implies that $\operatorname{supp} u \subseteq \overline{V} \subseteq O_a$.

On O_a, let us define $v := \frac{f_0}{|f_0|}$. This function is continuous on O_a. Since $\operatorname{supp} u \subseteq O_a$, the product function $u \cdot v$, defined as $\begin{cases} u(t)\,v(t), & \text{if } t \in O_a, \\ 0, & \text{if } t \in K \setminus O_a \end{cases}$, is continuous on K. Now, let us define

$$h(t) := \begin{cases} f_0(t) + u(t)\cdot\big(v(t) - f_0(t)\big), & t\in\Theta_a,\\[2mm] f_0(t), & t\in K\setminus\Theta_a.\end{cases}$$

Notice that h is continuous on K. Also, since we can write $h(t) = (1-u(t))f_0(t) + u(t)v(t)$ when $t\in\Theta_a$, $0\le u(t)\le 1$, $|f_0(t)|\le 1$ and $|v(t)|=1$, $\forall t\in K$, we have that $\|h\|\le 1$. Since $f_0\in S_{C(K)}$, there exists $\tilde{t}\in K$ such that $|f_0(\tilde{t})|=1$. Then, $\tilde{t}\in F_b$ and so $u(\tilde{t})=1$ and $|h(\tilde{t})|=|v(\tilde{t})|=1$. This means that $h\in S_{C(K)}$.

Now, let $A:=\{t\in K : |f_0(t)|\le b\}$. Then, A is closed and $K\setminus A$ is nonempty as $|f_0(\tilde{t})|=1>b$. If $t\in K\setminus A$, then $|f_0(t)|>b$, so $t\in F_b$, $u(t)=1$ and $h(t)=v(t)$, i.e., $|h(t)|=1$. Moreover, for every $t\in K$,

$$\begin{aligned} |h(t)-f_0(t)| &= u(t)\,|v(t)-f_0(t)| \\ &= u(t)\left|\frac{f_0(t)}{|f_0(t)|}-f_0(t)\right| \\ &= u(t)\big(1-|f_0(t)|\big) \le 1-a \end{aligned}$$

as $u(t)>0$ can occur only in Θ_a, where $|f_0(t)|>a$.

This proves (1), (2) and (3).

Next, let us choose a continuous function $G \in \mathcal{C}(K, [0,1])$ such that

$$G \equiv 1 \text{ on } A \quad \text{and} \quad G \equiv 0 \text{ on the set } \{t \in K : |f_0(t)| \geq c\}.$$

(this is once again Urysohn's lemma, since the sets $A = \{t \in K : |f_0(t)| \leq b\}$ and $\{t \in K : |f_0(t)| \geq c\}$ are disjoint closed sets).

Let $\mu : S \longrightarrow M(K)$ be w^*-continuous with $\|\mu\| = 1$.

By assumption, we have that

$$\mathrm{Re} \int_K f_0 \, d\mu(s_0) > 1 - \alpha$$

for some $\alpha > 0$. Let us put $\nu_0 = \mu(s_0)$, a complex regular Borel measure on K. Since $\|\mu\| = 1$, we get that $\|\nu_0\| \leq 1$. Also, by above, we also get that

$$1 - \alpha < \mathrm{Re} \int_K f_0 \, d\nu_0 \leq \int_K |f_0| \, d|\nu_0|. \quad (\#_1)$$

Let us write $W_c := \{t \in K : |f_0(t)| > c\}$. As $|f_0(t)| \leq c$, $\forall t \in K \setminus W_c$, $f_0 \in \mathcal{C}(K)$ with $\|f\| = 1$ and $|\nu_0|$ is a positive measure (for being the total variation of the complex measure ν_0), we can split the integral over W_c and $K \setminus W_c$ as follows

$$\int_K |f_0| \, d|\nu_0| = \int_{W_c} |f_0| \, d|\nu_0| + \int_{K \setminus W_c} |f_0| \, d|\nu_0|$$

$$\leq |\nu_0|(W_c) + c\, |\nu_0|(K \setminus W_c)$$

$$= |\nu_0|(W_c) + c\left[|\nu_0|(K) - |\nu_0|(W_c)\right]$$

$$= c|\nu_0|(K) + (1-c)\,|\nu_0|(W_c)$$

$$= c\,\|\nu_0\| + (1-c)\,|\nu_0|(W_c)$$

$$\leq c + (1-c)\,|\nu_0|(W_c).$$

Together with the previous estimate (#1), we obtain

$$|\nu_0|(W_c) > 1 - \frac{\alpha}{1-c}.$$

Since $G \equiv 0$ on W_c, we have that $1 - G \equiv 1$ on W_c. Also, since $0 \leq G \leq 1$, $1 - G \geq 0$ on K. So,

$$\int_K (1-G)\, d|\nu_0| = \int_{W_c} (1-G)\, d|\nu_0| + \int_{K\setminus W_c} (1-G)\, d|\nu_0|$$

$$= |\nu_0|(W_c) + \underbrace{\int_{K\setminus W_c} (1-G)\, d|\nu_0|}_{\geqslant 0}$$

$$\geqslant |\nu_0|(W_c)$$

$$> 1 - \frac{\alpha}{1-c}.$$

On the other hand, since $1-G \in \mathcal{C}(K)$ satisfies $1-G \geqslant 0$ on K and $\mu : S \longrightarrow M(K)$ is w^*-continuous, it follows, from the second part of Lemma 1, that $s \longmapsto \int_K (1-G)\, d|\mu(s)|$ is lower semicontinuous with respect to the original topology of S. From the previous inequality, there is an open neighborhood $\mathcal{U}_0$ of s_0 such that

$$(\#_2) \int_K (1-G)\, d|\mu(s)| > 1 - \frac{\alpha}{1-c} - \frac{\tau}{2}, \forall s \in \mathcal{U}_0.$$

Notice that since $\mu : S \longrightarrow M(K)$ is w^*-continuous, the scalar map $s \in S \longmapsto \int_K f_0 \, d\mu(s)$ is continuous.

By assumption, $\mathrm{Re} \int_K f_0 \, d\mu(s_0) > 1 - \alpha$. This means that the set

$$\left\{ s \in S : \mathrm{Re} \int_K f_0 \, d\mu(s) > 1 - \alpha \right\}$$

is an open neighborhood of s_0 in S. We can then shrink $\mathcal{U}_0$ if necessary such that

$$\int_K f_0 \, d\mu(s) > 1 - \alpha, \quad \forall s \in \mathcal{U}_0. \qquad (\#_3)$$

From $(\#_2)$ and $\|\mu(s)\| \leq 1$, $\forall s \in \mathcal{U}_0$, we obtain also that

Recall its def. in the statement of the lemma.

$$(\#_4) \quad \int_K G \, d|\mu(s)| < \frac{\alpha}{1 - c} + \frac{\tau}{2} < m, \quad \forall s \in \mathcal{U}_0.$$

Now, let us choose a nonempty open set $\mathcal{U}$ with $s_0 \in \mathcal{U}$ and $\overline{\mathcal{U}} \subseteq \mathcal{U}_0$ (S is compact Hausdorff $\Rightarrow$ normal $\Rightarrow$ regular). Once again by Urysohn's Lemma on S, there is $\varphi \in \mathcal{C}(S, [0,1])$ such that

$$\Psi \equiv 1 \text{ on } \overline{\mathcal{U}} \quad \text{and} \quad \operatorname{supp} \Psi \subseteq \mathcal{U}_0 .$$

For every $s \in S$, we define

$$d\mu_1(s) := (1 - \Psi(s)\, G)\, d\mu(s).$$

Then, $\forall\, \varphi \in \mathcal{C}(K)$, we have that

$$\int_K \varphi \, d\mu_1(s) = \int_K \varphi \,(1 - \Psi(s)\, G(t))\, d\mu(s)(t)$$

for every fixed $s \in S$. To prove that $\mu_1 : S \to M(K)$ is w^*-continuous, we must prove that, $\forall\, \varphi \in \mathcal{C}(K)$, the scalar map $s \longmapsto \int_K \varphi \, d\mu_1(s)$ is continuous. We have that

$$\int_K \varphi \, d\mu_1(s) = \int_K \varphi \,(1 - \Psi(s)\, G)\, d\mu(s)$$

$$= \int_K \varphi \, d\mu(s) - \Psi(s) \int_K \varphi G \, d\mu(s)$$

for a fixed $s \in S$. Now, $\varphi, G \in \mathcal{C}(K)$ and so $\varphi G \in \mathcal{C}(K)$. Since $\mu : S \to M(K)$ is w^*-continuous, both maps $s \mapsto \int_K \varphi \, d\mu(s)$ and $s \mapsto \int_K \varphi G \, d\mu(s)$ are continuous. Also, $s \mapsto \Psi(s)$ is continuous as $\Psi \in \mathcal{C}(S)$.

Therefore, $s \mapsto \int_K \varphi \, d\mu_1(s)$ is continuous, $\forall \varphi \in \mathcal{C}(K)$.

Notice also that, since $0 \leq 1 - \Psi(s) G \leq 1$, $\forall s \in S$, we have that $\|\mu_1\| \leq 1$. If $s \in \mathcal{U} \subseteq S$, then $\Psi(s) = 1$ and $G \equiv 1$ on $A \subseteq K$. So, $\forall s \in \mathcal{U}$, we have

$$|\mu_1(s)|(A) = \int_A (1 - \Psi(s) G) \, d|\mu(s)| = 0.$$

In words, this means that $\mu_1(s)$ gives no mass to A in total variation, $\forall s \in \mathcal{U}$. This gives (4) and (5).

Now, let us notice that if $s \notin \operatorname{supp} \Psi$, then $\mu_1(s) = \mu(s)$. If $s \in \operatorname{supp} \Psi$, then $s \in \mathcal{U}_0$ and using $(\#_4)$ we get

$$\|\mu_1(s) - \mu(s)\| = \int_K \Psi(s) G \, d|\mu(s)|$$

$$\leq \int G \, d|\mu(s)|$$

$$\overset{(\#_4)}{<} m.$$

Thus, $\|\mu_1 - \mu\| < m$. This gives (6).

Finally, for $s \in \mathcal{U}$, we have that

$$d\mu_1(s) = (1-G)\,d\mu(s).$$

Using $(\#_3)$, $(\#_4)$, $\|\mu(s)\| \leq 1$ and $\|h\| \leq 1$, we obtain, $\forall s \in \mathcal{U}$,

$$\begin{aligned}
\mathrm{Re}\int_K h\,d\mu_1(s) &= \mathrm{Re}\int_K h(1-G)\,d\mu(s) \\
&\geq \mathrm{Re}\int_K f_0\,d\mu(s) - \int_K |f_0 - h(1-G)|\,d|\mu(s)| \\
&\overset{(\#_3)}{>} 1-\alpha - \int_K \left(|f_0 - h| + G|h|\right) d|\mu(s)| \\
&\overset{(\#_4)}{\underset{(1)}{>}} 1-\alpha-(1-a)-m \\
&= a-\alpha-m \\
&= 1 - \underbrace{\left[(1-a)+\alpha+m\right]}_{=:d} \\
&= 1-d,
\end{aligned}$$

$\blacksquare$

and this shows (7).

The next lemma is an adaptation of Lemma 1.3 from your paper.

<u>Lemma 3</u>: Let K and S be compact Hausdorff spaces. Let $\mu : S \longrightarrow M(K)$ be w^*-continuous. Let $\mathcal{U} \subseteq S$ be nonempty and open. Let $A \subseteq K$ be closed with $K \setminus A \neq \emptyset$ and let $h \in \mathcal{C}(K)$ satisfy $\|h\| = 1$ and $|h(t)| = 1$, $\forall t \in K \setminus A$. Assume that $|\mu(s)|(A) = 0$, $\forall s \in \mathcal{U}$ and, for some $\delta > 0$, we also assume that

$$\mathrm{Re} \int_K h \, d\mu(s) > \|\mu\| - \delta, \ \forall s \in \mathcal{U}.$$

Fix $\frac{1}{2} < r < 1$. Then, there are a w^*-continuous map $\mu' : S \longrightarrow M(K)$ and a nonempty set $\mathcal{U}' \subseteq \mathcal{U}$ such that

(1) $|\mu'(s)(A)| = 0$, $\forall s \in \mathcal{U}'$,

(2) $\|\mu'\| \leq \|\mu\|$,

(3) $\mathrm{Re} \int_K h \, d\mu'(s) > \|\mu'\| - r\delta$, $\forall s \in \mathcal{U}'$,

(4) $\|\mu' - \mu\| \leq \sqrt{2\|\mu\|\delta} + 2\delta$.

<u>Proof</u>: If $\|\mu\| = 0$, then we take $\mu' = \mu$ and $\mathcal{U}' = \mathcal{U}$. Now, assume that $\|\mu\| > 0$ and let us write $a_0 := 1 - r$. So, we have that $0 < a_0 < 1/2$.

We distinguish two cases.

<u>Case 1</u>: Assume first that

$$\sup_{s \in \mathcal{U}} \|\mu(s)\| \leq \|\mu\| - a_0 \delta.$$

Let $s_0 \in \mathcal{U}$ and $t_0 \in K \setminus A$. By assumption, $|h(t_0)| = 1$. So, denoting by δ_{t_0} the Dirac measure on K at t_0,

$$\int_K h \, d(\overline{h(t_0)} \, \delta_{t_0}) = \overline{h(t_0)} \cdot h(t_0) = |h(t_0)|^2 = 1.$$

Choose $\psi \in \mathcal{C}(S, [0,1])$ with $\psi(s_0) = 1$ and $\operatorname{supp} \psi \subseteq \mathcal{U}$. Let us define

$$\mu'(s) := \mu(s) + a_0 \cdot \delta \cdot \psi(s) \cdot \overline{h(t_0)} \cdot \delta_{t_0}$$

for every $s \in S$. Notice that this map is w^*-continuous as, $\forall \varphi \in \mathcal{C}(K)$,

$$\int_K \varphi \, d\mu'(s) = \int_K \varphi \, d\mu(s) + a_0 \cdot \delta \cdot \psi(s) \overline{h(t_0)} \cdot \varphi(t_0)$$

for a fixed $s \in S$. The first term is continuous because $\mu : S \to M(K)$ is w^*-continuous and the second is also continuous because $\psi \in \mathcal{C}(S)$. Hence $s \mapsto \int_K \varphi \, d\mu(s)$ is continuous $\forall \varphi \in \mathcal{C}(K)$ and so μ' is w^*-continuous.

Also, if $s \notin \mathcal{U}$, then $\mu'(s) = \mu(s)$ and if $s \in \mathcal{U}$, then

$$\|\mu'(s)\| \leq \|\mu(s)\| + \|a_0 \delta \psi(s) \overline{h(t_0)} \delta_{t_0}\|.$$

Since $t_0 \in K \setminus A$, we have that $|\overline{h(t_0)}| = |h(t_0)| = 1$. Also, $\|\delta_{t_0}\| = 1$. Since $0 \leq \psi \leq 1$, we get

$$\|\mu'(s)\| \leq \|\mu(s)\| + a_0 \cdot \delta, \ \forall s \in \mathcal{U}.$$

Since we are assuming $\sup_{s \in \mathcal{U}} \|\mu(s)\| \leq \|\mu\| - a_0 \cdot \delta$, we get that

$$\|\mu'(s)\| \leq \|\mu\| - a_0 \cdot \delta + a_0 \cdot \delta = \|\mu\|, \ \forall s \in \mathcal{U}.$$

This shows that $\|\mu'\| = \sup_{s \in S} \|\mu'(s)\| \leq \|\mu\|$.

Now, by assumption $|\mu(s)|(A) = 0, \forall u \in S$. Since $t_0 \in K \setminus A$, we get that $|\mu'(s)|(A) = 0, \forall s \in \mathcal{U}$.

Notice also that at $s_0 \in S$, we have that

$$\mathrm{Re}\int_K h\, d\mu'(s_0) = \mathrm{Re}\int_K h\, d\mu(s_0) + a_0 \cdot \delta$$

$$\overset{\text{By assumption}}{>} \|\mu\| - \delta + a_0 \cdot \delta$$

$$= \|\mu\| - \delta + (1-r)\cdot \delta$$

$$= \|\mu\| - r\delta$$

$$\geq \|\mu'\| - r\delta.$$

By scalar continuity, the same strict inequality holds on some neighborhood $\mathcal{U}' \subseteq \mathcal{U}$ of s_0. Finally,

$$\|\mu' - \mu\| \leq a_0 \cdot \delta \leq \sqrt{2\|\mu\|\delta} + 2\delta.$$

This proves (1), (2), (3) and (4) in Case 1.

Now we consider the second case.

<u>Case 2</u>: Let us assume now that

$$\sup_{s\in\mathcal{U}} \|\mu(s)\| > \|\mu\| - a_0\,\delta.$$

We can then pick $s_1 \in \mathcal{U}$ to be such that

$$\|\mu(s_1)\| > \|\mu\| - a_0 \cdot \delta. \qquad (\#_5)$$

Since $\mu(s_1) \in M(K)$ is a <u>complex</u> regular Borel measure on K, we replace it by its total variation measure. Define $\sigma := |\mu(s_1)|$. So, σ is a positive finite regular Borel measure on K and its total mass is $\sigma(K) = |\mu(s_1)|(K) = \|\mu(s_1)\|$. Now, the complex measure $\mu(s_1)$ is absolutely continuous with respect to σ since if $\sigma(E) = |\mu(s_1)|(E) = 0$, then automatically, $\mu(s_1)(E) = 0$. Therefore, by the Radon-Nikodym theorem, there is a σ-measurable function $\theta : K \to \mathbb{C}$ such that $d\mu(s_1) = \theta\, d\sigma$ with $|\theta| = 1$ σ-a.e..

Now, since $s_1 \in \mathcal{U}$, by assumption, $|\mu(s_1)|(A) = 0$ and so $\sigma(A) = 0$. Since $|h| = 1$ on $K \setminus A$, the function $g := h\theta$ has complex modulus one σ-almost everywhere.

The hypothesis implies, as $\|\mu(s_1)\| \leq \|\mu\|$,

$$\|\mu(s_1)\| - \operatorname{Re} \int_K h \, d\mu(s_1) < \delta. \qquad (\#_6)$$

Now, choose $\gamma > 0$ to be such that

$$\gamma < (2r-1)\delta. \qquad (\#_7)$$

↳ recall r from the assumptions

Since $\mathscr{C}(K)$ is dense in $L_1(\sigma)$ [Rudin's book, Theorem 3.14], there is $u \in \mathscr{C}(K)$ such that

$$\int_K |u - \overline{\xi}| \, d\sigma < \gamma.$$

Define $P : \mathbb{C} \longrightarrow \overline{\mathbb{D}}$ by

$$P(z) = \begin{cases} z, & \text{if } |z| \leq 1, \\ \dfrac{z}{|z|}, & \text{if } |z| > 1. \end{cases}$$

Why we consider P? Well, $u \in \mathscr{C}(K)$ above might not satisfy $\|u\|_\infty \leq 1$, so we modify u by projecting its values into the closed unit disk. That is, we consider the function $q := P \circ u \in \mathscr{C}(K)$. Then, $\|q\|_\infty \leq 1$. Now, since $\overline{\xi} \in \mathbb{D}$ σ-almost everywhere, we have that

$$|P(u(t)) - \overline{\xi(t)}| \le |u(t) - \overline{\xi(t)}|,$$

$\forall\, t \in K$ σ-a.e. So,

$$\int_K |q - \bar{\xi}|\, d\sigma \le \int_K |u - \bar{\xi}|\, d\sigma < \gamma. \qquad (\#_8)$$

Once again using that $|\xi| = 1$ σ-a.e., we also get

$$\int_K |q\xi - 1|\, d\sigma = \int_K |(q - \bar{\xi})\xi|\, d\sigma < \gamma \qquad (\#_9)$$

by using $(\#_8)$. Therefore,

σ is a positive measure

$$\begin{aligned}
\operatorname{Re} \int_K h\, q\, d\mu(s_1) &= \int_K \operatorname{Re}(q\, h\, \theta)\, d\sigma \\
&= \int_K \operatorname{Re}(q\, \xi)\, d\sigma \\
&\ge \underbrace{\int_K 1\, d\sigma}_{\substack{=|\mu(s_1)|(K) \\ = \|\mu(s_1)\|}} - \int_K |q - \bar{\xi}|\, d\sigma \\
&\overset{(\#_9)}{>} \|\mu(s_1)\| - \gamma \\
&\overset{(\#_5)}{>} \|\mu\| - a_0 \cdot \delta - \gamma \\
&= \|\mu\| - (1 - r) \cdot \delta - \gamma
\end{aligned}$$

$$\overset{(\#_7)}{>} \|\mu\| - (1-r)\delta - (2r-1)\delta$$

$$= \|\mu\| - \delta + r\delta - 2r\delta + \delta$$

$$= \|\mu\| - r\delta. \quad (\widetilde{\#})$$

We need now to estimate $\int_K |q-1|\, d\sigma$. Since $|\mathfrak{z}| = 1$ σ-a.e., we also have that

$$|\mathfrak{z} - 1|^2 = 2(1 - \operatorname{Re} \mathfrak{z})$$

σ-almost everywhere. Using the Cauchy inequality and $(\#_6)$, we get

$$\int_K |\mathfrak{z} - 1|\, d\sigma \leq \left(\int_K |\mathfrak{z} - 1|^2 d\sigma \right)^{1/2} \sigma(K)^{1/2}$$

$$= \left(2 \int (1 - \operatorname{Re} \mathfrak{z})\, d\sigma \right)^{1/2} \|\mu(S_1)\|^{1/2}$$

$$= \left(2 \left[\|\mu(S_1)\| - \operatorname{Re} \int h\, d\mu(S_1) \right] \right)^{1/2} \|\mu(S_1)\|^{1/2}$$

$$< \sqrt{2 \|\mu\| \cdot \delta}. \quad (\#_8)$$

Puting this together with [redacted] gives us the following inequality

$$\int_K |q-1|\, d\sigma \leq \int_K |q-\bar{g}|\, d\sigma + \int_K |\bar{g}-1|\, d\sigma$$

$$\overset{(\#8)}{<} \gamma + \sqrt{2\|\mu\|\,\delta}\,. \qquad (\#_{10})$$

Let us set

$$\varphi := \frac{|q-1|}{2} \in \mathcal{C}(K).$$

Then, $0 \leq \varphi \leq 1$ and by Lemma 1 the map

$$s \longmapsto \int_K (1-\varphi)\, d|\mu(s)|$$

is lower semicontinuous as $\mu : S \longrightarrow M(K)$ is w^*-continuous. Let us choose $\eta > 0$ so small that

$$2a_0\delta + \gamma + 2\eta < 2\delta \qquad (\#_{11})$$

(which is possible provided we chose γ small enough as before). There exists an open neighborhood $\mathcal{U}_0 \subseteq \mathcal{U}$ of S_1 such that, $\forall s \in \mathcal{U}_0$, we have

$$\int_K (1-\varphi)\, d|\mu(s)| > \int_K (1-\varphi)\, d\sigma - \eta.$$

Therefore, $\forall s \in \mathcal{U}_0$,

$$
\begin{aligned}
\int_K \varphi \, d|\mu(s)| &= \|\mu(s)\| - \int_K (1-\varphi)\, d|\mu(s)| \\
&\leq \|\mu\| - \int_K (1-\varphi)\, d|\mu(s)| \\
&< \|\mu\| - \int_K (1-\varphi)\, d\sigma + \eta \\
&= \|\mu\| - \|\mu(s_1)\| + \int_K \varphi \, d\sigma + \eta .
\end{aligned}
$$

Hence, for $s \in \mathcal{U}_0$, we obtain

$$
\begin{aligned}
\int_K |q-1| \, d|\mu(s)| &\overset{\text{def.}}{=} 2 \int_K \varphi \, d|\mu(s)| \\
&< 2 \left(\|\mu\| - \|\mu(s_1)\| + \int_K \varphi \, d\sigma + \eta \right) \\
&\overset{(\#5)}{<} 2 a_0 \delta + \int_K |q-1| \, d\sigma + 2\eta \\
&\overset{(\#10)}{<} \sqrt{2\|\mu\|\delta} + 2a_0\delta + \delta + 2\eta \\
&\overset{(\#11)}{<} \sqrt{2\|\mu\|\delta} + 2\delta . \qquad (\widetilde{\#})
\end{aligned}
$$

Once again, choose $\psi \in \mathcal{C}(S, [0,1])$ to be such that $\psi(s_1) = 1$ and $\operatorname{supp} \psi \subseteq \mathcal{U}_0$. Define

$$d\mu'(s) := \big((1 - \psi(s)) + \psi(s) q \big) \, d\mu(s)$$

for every $s \in S$. Notice first that the coefficient $(1 - \psi(s)) + \psi(s) q$ is a convex combination of 1 and an element in $\mathbb{D}$. This means that we get $\|\mu'\| \leq \|\mu\|$. Now, let us see that $\mu' : S \to M(K)$ is w^*-continuous. For this, we must prove that, $\forall \phi \in \mathcal{C}(K)$, the scalar function $s \mapsto \int_K \phi \, d\mu'(s)$ is continuous on S. Indeed, $\forall \phi \in \mathcal{C}(K)$, for a fixed $s \in S$,

$$\int_K \phi \, d\mu'(s) = (1 - \psi(s)) \int_K \phi \, d\mu(s) + \psi(s) \int_K \phi q \, d\mu(s).$$

Since $s \mapsto \psi(s)$ is continuous as $\psi \in \mathcal{C}(S)$, $s \mapsto \int_K \phi \, d\mu(s)$ is also continuous because $\mu : S \to M(K)$ is w^*-continuous and $\phi \in \mathcal{C}(K)$ and $\phi q \in \mathcal{C}(K)$ because $\phi \in \mathcal{C}(K)$ and $q \in \mathcal{C}(K)$, and again by the w^*-continuity of μ, $s \mapsto \int_K \phi q \, d\mu(s)$ is continuous, we have that $s \mapsto \int_K \phi \, d\mu'(s)$ is continuous on S and $\mu' : S \to M(K)$ is w^*-continuous.

Moreover, since $|\mu(s)|(A)=0$ if $s\in\mathcal{U}_0$, we have $|\mu'(s)|(A)=0, \ \forall s\in\mathcal{U}_0$.

At $s_1\in S$, we have that

$$\mathrm{Re}\int_K h\,d\mu'(s_1) = \mathrm{Re}\int_K h q\,d\mu(s_1) \overset{(\tilde{\#})}{>} \|\mu\| - r\delta$$

$$\geq \|\mu'\| - r\delta.$$

By scalar continuity, this strict inequality holds on a nonempty open set $\mathcal{U}'\subseteq\mathcal{U}_0$ containing s_1.

Finally, for $s\in\mathcal{U}_0$, since $d(\mu'(s)-\mu(s)) = \psi(s)(q-1)$, we have

$$\|\mu'(s)-\mu(s)\| \leq \int_K |q-1|\,d|\mu(s)|$$

$$\overset{(\tilde{\tilde{\#}})}{<} \sqrt{2\|\mu\|\delta} + 2\delta$$

while $\mu'(s)=\mu(s)$ outside of $\mathrm{supp}(\psi)\subseteq\mathcal{U}_0$. Taking the supremum over $s\in S$, we get (4). This proves Case 2 and, consequently, the proof. ∎

Now, we're ready for our main result.

$\boxed{\text{Result}}$ Consider $\mathcal{C}(K)$ and $\mathcal{C}(S)$ as complex Banach spaces with the supremum norm, where K and S are compact Hausdorff spaces.

Fact: $(\mathcal{C}(K), \mathcal{C}(S))$ has the Bishop-Phelps-Bollobás property for operators. In other words, for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $T_0 \in S_{\mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))}$ and $f_0 \in S_{\mathcal{C}(K)}$ satisfy

$$\| T_0(f_0) \| > 1 - \eta(\varepsilon),$$

there are $T \in S_{\mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))}$ and $h \in S_{\mathcal{C}(K)}$ such that

$$\| T(h) \| = 1, \quad \| T - T_0 \| < \varepsilon \quad \text{and} \quad \| h - f_0 \| < \varepsilon.$$

In particular, we get the density, i.e.,

$$\overline{NA(\mathcal{C}(K), \mathcal{C}(S))} = \mathcal{L}(\mathcal{C}(K), \mathcal{C}(S)).$$

Proof: Fix $0<\varepsilon<1$. Let us define all the parameters we need in this proof, which are quite a few as one can imagine.

Choose ~~a~~ some number $\frac{1}{2}<r<1$. Choose also $\kappa>0$ (this is kappa!) so small such that

$$\frac{\sqrt{2\kappa}}{1-\sqrt{r}}+\frac{2\kappa}{1-r}<\frac{\varepsilon}{4}. \qquad (\#_{12})$$

Choose $0<a<1$ close to 1 so that

$$1-a<\min\left\{\varepsilon,\frac{\kappa}{3}\right\}. \qquad (\#_{13})$$

Then, choose numbers $b,c\in\mathbb{R}$ with

$$a<b<c<1. \qquad (\#_{14})$$

Choose $\tau>0$ so small such that

$$\tau<\min\left\{\frac{\kappa}{3},\frac{\varepsilon}{16}\right\}. \qquad (\#_{15})$$

Finally, let us choose $\eta>0$ small enough so that

$$\eta\left(1+\frac{1}{1-c}\right) < \frac{k}{3} \qquad (\#_{16})$$

and

$$\frac{\eta}{1-c} + \tau < \frac{\varepsilon}{8}. \qquad (\#_{17})$$

This η will be the one for the Bishop-Phelps-Bollobás property for the given $\varepsilon > 0$.

Now we can finally start the proof. Let us take $T_0 \in \mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))$ with $\|T_0\| = 1$ and $f_0 \in S_{\mathcal{C}(K)}$ to be such that

$$\|T_0(f_0)\| > 1 - \eta.$$

By the representation theorem for operators from $\mathcal{C}(K)$ into $\mathcal{C}(S)$ [Dunford and Schwartz, Theo 1 pg 490], there exists a w^*-continuous map $\mu_0 : S \longrightarrow M(K)$ such that

$$T_0 f(s) = \int_K f \, d\mu_0(s), \quad \forall f \in \mathcal{C}(K), \forall s \in S.$$

Also, $\|\mu_0\| = \|T_0\| = 1$.

Since $T_0(f_0) \in \mathcal{E}(S)$ and its norm is longer than $1-\eta$, there exists $s_0 \in S$ such that

$$\left| \int_K f_0 \, d\mu_0(s_0) \right| > 1-\eta .$$

Now, take $\lambda \in \mathbb{T}$ such that, after setting $\mu := \lambda\mu_0$, one gets

$$\operatorname{Re} \int_K f_0 \, d\mu(s_0) > 1-\eta .$$

We have that $\|\mu\| = 1$.

Let us apply Lemma 1 to this μ, to f_0, to s_0 and to the parameters we've picked before, that is, to a, b, c, τ and consider the α from Lemma 1 to be $\alpha = \eta$. We the obtain $h \in S_{\mathcal{E}(K)}$, a closed set $A \subseteq K$, a non-empty open set $\mathcal{U} \subseteq S$ and a w^*-continuous map $\mu_1 : S \longrightarrow M(K)$ such that

$$\|h - f_0\| \leq 1 - \alpha \overset{(\#_{13})}{<} \varepsilon , \quad (\#_{18})$$

$|h(t)| = 1$, $\forall t \in K \setminus A$, the set $K \setminus A \neq \emptyset$ is non-empty, $|\mu_1(s)|(A) = 0$, $\forall s \in \mathcal{U}$, $\|\mu_1\| \leq 1$,

$$\|\mu_1 - \mu\| < m := \frac{\eta}{1-c} + \tau \qquad (\#_{19})$$

and

$$\mathrm{Re} \int_K h \, d\mu_1(s) > 1 - d, \ \forall s \in \mathcal{U}, \qquad (\#_{20})$$

where $d := (1-a) + \eta + m$.

From $(\#_{13})$, $(\#_{16})$ and $(\#_{15})$, we have

$$(\#_{21}) \quad d = (1-a) + \eta + \frac{\eta}{1-c} + \tau < \frac{\kappa}{3} + \frac{\kappa}{3} + \frac{\kappa}{3} = \kappa.$$

Also, by $(\#_{17})$,

$$m = \frac{\eta}{1-c} + \tau < \frac{\varepsilon}{8}. \qquad (\#_{22})$$

Now, since $\|\mu_1\| \leq 1$, $(\#_{20})$ implies that

$$\mathrm{Re} \int_K h \, d\mu_1(s) > \|\mu_1\| - d, \ \forall s \in \mathcal{U}.$$

We now iterate Lemma 3. Let us do this carefully. Set $\mu^{(0)} := \mu_1$, $\mathcal{U}_0 := \mathcal{U}$ and $\delta_n := r^n d$, for $n = 0, 1, 2, 3, \ldots$

Inductively, suppose that $\mu^{(n)} : S \to M(K)$ is w^*-continuous, $\mathcal{U}_n \subseteq S$ is non-empty and open and

$$|\mu^{(n)}(s)|(A) = 0, \ \forall s \in \mathcal{U}_n,$$

$$\|\mu^{(n)}\| \leq 1$$

and

$$\operatorname{Re} \int_K h \, d\mu^{(n)}(s) > \|\mu^{(n)}\| - \delta_n, \ \forall s \in \mathcal{U}_n.$$

Applying Lemma 3 to $\mu^{(n)}$, $\mathcal{U}_n$, A, h and δ_n, we obtain a w^*-continuous map $\mu^{(n+1)} : S \to M(K)$ and a non-empty open set $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n \subseteq S$ such that

(1) $|\mu^{(n+1)}(s)|(A) = 0, \ \forall s \in \mathcal{U}_{n+1}$,

(2) $\|\mu^{(n+1)}\| \leq \|\mu^{(n)}\| \leq 1$,

(3) $\operatorname{Re} \int_K h \, d\mu^{(n+1)}(s) > \|\mu^{(n+1)}\| - r\delta_n$

$$= \|\mu^{(n+1)}\| - \delta_{n+1}, \ \forall s \in \mathcal{U}_{n+1}$$

and

(4) $\|\mu^{(n+1)} - \mu^{(n)}\| \leq \sqrt{2\|\mu^{(n)}\|\delta_n} + 2\delta_n$

$$\leq \sqrt{2r^n d} + 2r^n d.$$

Then, the induction follows.

First, notice that

$$\sum_{n=0}^{+\infty} \|\mu^{(n+1)} - \mu^{(n)}\| \overset{(4)}{\leq} \sum_{n=0}^{+\infty} \left(\sqrt{2r^n d} + 2r^n d\right)$$

$$\overset{(\#21)}{<} \sum_{n=0}^{+\infty} \left(\sqrt{2r^n K} + 2r^n K\right).$$

The right hand side sum is finite as it is a geometric series. In fact,

$$\sum_{n=0}^{+\infty} \sqrt{2r^n K} = \sqrt{2K} \sum_{n=0}^{+\infty} r^{n/2} = \frac{\sqrt{2K}}{1-\sqrt{r}}$$

and

$$\sum_{n=0}^{+\infty} 2r^n K = 2K \sum_{n=0}^{+\infty} r^n = \frac{2K}{1-r}.$$

Thus, we have that

$$\sum_{n=0}^{+\infty} \|\mu^{(n+1)} - \mu^{(n)}\| < \frac{\sqrt{2K}}{1-\sqrt{r}} + \frac{2K}{1-r} \overset{(\#12)}{<} \frac{\varepsilon}{4}.$$

This means that $(\mu^{(n)})_{n=0}^{+\infty}$ is a Cauchy sequence in the Banach space of w*-continuous maps from S into $M(K)$ endowed with the norm $\|\nu\| = \sup_{s\in S}\|\nu(s)\|$. So, there exists a map $\nu : S \longrightarrow M(K)$ with

$$\|\nu - \mu_1\| < \frac{\varepsilon}{4}. \qquad (\#_{23})$$

Observe that ν is w*-continuous as, $\forall\, \phi \in \mathcal{C}(K)$, the scalar functions $s \longmapsto \int_K \phi \, d\mu^{(n)}(s)$ converge uniformly to $s \longmapsto \int_K \phi \, d\nu(s)$ since

$$\left| \int_K \phi \, d\left(\nu(s) - \mu^{(n)}(s)\right) \right| \leq \|\phi\| \cdot \|\nu - \mu^{(n)}\|$$

and a uniform limit of continuous scalar functions is continuous. This shows that ν is w*-continuous.

Now, notice that

$$\|\nu - \mu\| \leq \|\nu - \mu_1\| + \|\mu_1 - \mu\| \underset{(\#_{19})}{\overset{(\#_{23})}{<}} \frac{\varepsilon}{4} + m \overset{(\#_{22})}{<} \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{8}. \qquad (\#_{24})$$

Notice also that, from the above inequality and the fact that $\|\mu\| = 1$, we have that ν is non-zero.

<u>Claim 1</u>: The operator represented by ν attains its norm at h.

<u>PROOF</u>: For each n, let us choose $s_n \in U_n$. Then,

$$\operatorname{Re} \int_K h \, d\mu^{(n)}(s) > \|\mu^{(n)}\| - \delta_n .$$

Thus,

$$\sup_{s \in S} \left| \int_K h \, d\nu(s) \right| \geq \operatorname{Re} \int_K h \, d\nu(s_n)$$

$$\geq \operatorname{Re} \int_K h \, d\mu^{(n)}(s_n) - \|\nu - \mu^{(n)}\|$$

$$> \|\mu^{(n)}\| - \delta_n - \|\nu - \mu^{(n)}\| .$$

Letting $n \to +\infty$, and using that $\delta_n \to 0$ and $\mu^{(n)} \longrightarrow \nu$ in norm as $n \to +\infty$, we get

$$\sup_{s \in S} \left| \int_K h \, d\nu(s) \right| \geq \|\nu\| .$$

On the other hand, since $h \in S_{\mathcal{C}(K)}$, we have

$$\left| \int_K h \, d\nu(s) \right| \leq \|\nu(s)\| \leq \|\nu\|, \ \forall s \in S.$$

This shows that

$$\sup_{s \in S} \left| \int_K h \, d\nu(s) \right| = \|\nu\|. \qquad (\#_{25})$$

Recall that $\lambda \in \mathbb{T}$ is the one such that $\mu = \lambda \mu_0$. Now, let us define $\tilde{\mu} : S \longrightarrow M(K)$ by

$$\tilde{\mu}(s) := \overline{\lambda} \cdot \frac{\nu(s)}{\|\nu\|}, \ \forall s \in S.$$

Since ν is w^*-continuous, so is $\tilde{\mu}$. Also,

$$\|\tilde{\mu}\| = \sup_{s \in S} \|\tilde{\mu}(s)\| = \frac{1}{\|\nu\|} \sup_{s \in S} \|\nu(s)\| = 1.$$

Now, define $T \in \mathcal{L}(\mathcal{C}(K), \mathcal{C}(S))$ the operator represented by $\tilde{\mu}$, that is,

$$Tf(s) = \int_K f \, d\tilde{\mu}(s), \ \forall f \in \mathcal{C}(K), \forall s \in S.$$

Then, $\|T\| = \|\tilde{\mu}\| = 1$. Also, by $(\#_{25})$,

$$\|T(h)\| = \frac{1}{\|\nu\|} \sup_{s \in S} \left| \int_K h \, d\nu(s) \right| = 1. \quad (\tilde{\#})$$

$\square$

<u>Claim 2</u>: $\|T - T_0\| < \varepsilon$.

<u>PROOF</u>: By the representation theorem, we have that

$$\|T - T_0\| = \|\tilde{\mu} - \mu_0\| = \left\| \bar{\lambda} \cdot \frac{v}{\|v\|} - \mu_0 \right\|$$

$$= |\lambda| \cdot \left\| \bar{\lambda} \cdot \frac{v}{\|v\|} - \mu_0 \right\|$$

$$= \left\| \frac{v}{\|v\|} - \lambda \mu_0 \right\|$$

recall that $\mu = \lambda \mu_0$

$$= \left\| \frac{v}{\|v\|} - \mu \right\|$$

$$\leq \left\| \frac{v}{\|v\|} - v \right\| + \|v - \mu\|$$

$$= \big| 1 - \|v\| \big| + \|v - \mu\|$$

$$= \big| \|v\| - \|\mu\| \big| + \|v - \mu\|$$

$$\leq 2 \|v - \mu\|$$

$$\overset{(\#24)}{<} \frac{3\varepsilon}{4} < \varepsilon. \qquad \square$$

We already have by ($\#_{18}$) that $\|h - f_0\| < \varepsilon$. Since T attains its norm at h by ($\tilde{\#}$) and $\|T - T_0\| < \varepsilon$ by Claim 2, we can conclude that $(\mathcal{C}(K), \mathcal{C}(S))$ satisfies the Bishop-Phelps-Bollobás property for operators. $\blacksquare$