Theo: Let X, Y be finite-dimensional spaces. Then, the pair (X,Y) has the BPBp for operators. For every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $T \in S_{\mathcal{L}(X,Y)}$, there is $R \in S_{\mathcal{L}(X,Y)}$ satisfying ~~such that~~ $\|R-T\| < \varepsilon$, and such that whenever $x \in S_X$ satisfies $\|Tx\| > 1-\delta$, there is $\tilde{x} \in S_X$ with $\|R\tilde{x}\| = 1$ and $\|\tilde{x} - x\| < \varepsilon$.

Proof: Since X, Y are fin-dim, the operator space $\mathcal{L}(X,Y)$ is fin-dim. Hence, its unit sphere $S_{\mathcal{L}(X,Y)}$ is compact. For each $A \in S_{\mathcal{L}(X,Y)}$, define its norm-attainment set by

$$M_A := \{x \in S_X : \|Ax\| = 1\}.$$

Since X is finite-dim, S_X is also ~~again~~ compact. The function $x \mapsto \|Ax\|$ is continuous on S_X, so it attains its maximum. Since $\|A\| = 1$, we then have that $M_A \neq \emptyset$. Moreover, M_A is compact.

Now, consider, for a fixed $A \in S_{\mathcal{L}(X,Y)}$, the compact set

$$K_A := \{x \in S_X : \operatorname{dist}(x, M_A) \geq \varepsilon\}$$

This is the set of unit vectors which are at least ε away from every norm-attaining point of A.

Being a closed subset of S_X, K_A is also compact. Also, $K_A \cap M_A = \emptyset$. Therefore, since $x \mapsto \|Ax\|$ is continuous and equals 1 exactly on M_A, we have that

$$\max_{x \in K_A} \|Ax\| < 1.$$

Define $m_A := \max_{x \in K_A} \|Ax\|$. Thus $m_A < 1$. Set

$$\eta_A := \frac{1 - m_A}{4} > 0.$$

<u>Claim</u>: If an operator $B \in S_{\mathcal{L}(X,Y)}$ is sufficiently closed to A, then every point at which B nearly attains its norm is close to a norm-attaining point of A.

<u>Proof</u>: Indeed, suppose that $\|B - A\| < \eta_A$, and $x \in S_X$ satisfies $\|Bx\| > 1 - \eta_A$. Then,

$$\|Ax\| = \|Bx\| - \|(B-A)x\| > 1 - \eta_A - \eta_A = 1 - 2\eta_A$$

By definition of η_A,

$$1 - 2\eta_A = 1 - \frac{1 - m_A}{2} = \frac{1 + m_A}{2}.$$

Since $m_A < 1$, $\frac{1 + m_A}{2} > m_A$, hence $\|Ax\| > m_A$. But m_A is the max over all unit vectors at distance at least ε from M_A. Therefore, $x \notin K_A$ and then $\operatorname{dist}(x, M_A) < \varepsilon$; thus, $\exists\, \tilde{x} \in M_A : \|x - \tilde{x}\| < \varepsilon$. Since $\tilde{x} \in M_A$, $\|A\tilde{x}\| = 1$. So, A itself is a norm-one operator attaining its norm at $\tilde{x}$ and $\|x - \tilde{x}\| < \varepsilon$.

Now we ~~gen~~ globalize this local argument uniformly over all norm-one operators.

For each $A \in S_R(X,Y)$, let us define

$$r_A := \min\left\{\eta_A, \frac{\varepsilon}{2}\right\}.$$

The balls

$$B(A, r_A) := \{B \in S_R(X,Y) : \|B - A\| < r_A\}$$

form an open ~~cove~~ cover of the compact set $S_R(X,Y)$.

Therefore there exist finitely many operators $A_1, \dots, A_N \in S_R(X,Y)$ such that

$$S_R(X,Y) \subseteq \bigcup_{j=1}^{N} B(A_j, r_{A_j}).$$

For each $j = 1, \dots, N$, write $\eta_j := \eta_{A_j}$ and $r_j = r_{A_j}$. Define

$$\delta := \min_{1 \le j \le N} \eta_j.$$

Since $\eta_j > 0$, we have that $\delta > 0$. Notice that δ depends only on X, Y, ε and on the finite cover chosen from the compact set $S_R(X,Y)$. It does not depend on the operator T.

Now take an arbitrary operator $T \in S_{\mathcal{L}(X,Y)}$. By the finite covering, there is $j \in \{1, \dots, N\}$ such that $\|T - A_j\| < r_j$. Set $R := A_j$. Then, $R \in S_{\mathcal{L}(X,Y)}$ and

$$\|R - T\| = \|A_j - T\| < r_j \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now let $x \in S_X$ satisfy $\|Tx\| > 1 - \delta$. Since $\delta \leq \eta_j$, we have $\|Tx\| > 1 - \eta_j$. Also, $\|T - A_j\| < r_j \leq \eta_j$. By the local claim applied to $A = A_j$ and $B = T$, it follows that $\mathrm{dist}(x, M_{A_j}) < \varepsilon$. Therefore, there is $\tilde{x} \in M_{A_j}$ such that $\|x - \tilde{x}\| < \varepsilon$. Since $R = A_j$, this means $\|R\tilde{x}\| = 1$. Thus R is a norm-one operator, close to T, and every point where T almost attains its norm is close to a point where R attains its norm. $\blacksquare$