Lemma: Let K be a compact topological space and let $f: K \to \mathbb{R}$ be a continuous function. Suppose that $f(t) < 1$, $\forall t \in K$. Then, there is a number $c < 1$ such that $f(t) \leq c$, $\forall t \in K$.

Proof: If $K = \emptyset$, nothing to prove. Assume K non-empty. Fix $t \in K$. Since $f(t) < 1$, choose a number c_t such that $f(t) < c_t < 1$. By continuity of f, there is an open ngh U_t of t such that $f(s) < c_t$, $\forall s \in U_t \cap K$. The family $\{U_t : t \in K\}$ is an open cover for K. Since K is compact, there are $t_1, \ldots, t_n \in K$ such that $K \subseteq U_{t_1} \cup \ldots \cup U_{t_n}$. Set $c := \max\{c_{t_1}, \ldots, c_{t_n}\}$. Since each c_{t_j} is < 1, $c < 1$. If $s \in K$, then $s \in U_{t_j}$ for some $j \in \{1, \ldots, n\}$ and hence $f(s) \leq c_{t_j} \leq c$. Therefore, $f(t) \leq c$, $\forall t \in K$. $\blacksquare$

Prop 1: Let X be Banach. Suppose that X^* has the w^*-KK. Then, X is SSD (= $(X, \mathbb{K})$ has the Lp,p).

Proof: Fix $x_0 \in S_X$ and $\varepsilon > 0$. We shall find a number $\eta(\varepsilon, x_0) > 0$ satisfying the conditions of the Lp,p for $(X, \mathbb{K})$. Let

$$M(x_0) := \{x^* \in S_{X^*} : |x^*(x_0)| = 1\}$$

which is non-empty by H-B. In fact, we may write $M(x_0) = \{x^* \in B_{X^*} : |x^*(x_0)| = 1\}$ as $\|x_0\| = 1$. Hence, $M(x_0)$ is a w^*-closed in B_{X^*} and, therefore, w^*-compact by Banach-Alaoglu.

By assumption, X^* has the w^*-KK which says that the w^*-topology and the $\|\cdot\|$-norm topology coincide in S_{X^*}. Therefore, for each $u^* \in M(x_0)$, there is a w^*-open ngh V_{u^*} of u^* such that

$$V_{u^*} \cap S_{X^*} \subseteq \{x^* \in S_{X^*} : \|x^* - u^*\| < \varepsilon\}$$

Set

$$V := \bigcup_{u^* \in M(x_0)} V_{u^*}.$$

Then, V is a w^*-star open ngh of M and

$$V \cap S_{X^*} \subseteq \{x^* \in S_{X^*} : \operatorname{dist}(x^*, M) < \varepsilon\}.$$

Now put $K := B_{X^*} \setminus V$. Notice that K is w^*-compact as B_{X^*} is w^*-compact and V is w^*-star open. Define $f : K \to \mathbb{R}$ by $f(x^*) := |x^*(x_0)|$. This function is w^*-continuous on K. For every $x^* \in K$, we have $f(x^*) < 1$.

Indeed, if $x^* \in B_{X^*}$ and $|x^*(x_0)| = 1$, then $\|x^*\| = 1$ and $x^* \in M(x_0)$. Since $M(x_0) \subseteq V$, such a point does not belong to $K = B_{X^*} \setminus V$.

Thus all points of K satisfy $f(x^*) < 1$.

By the lemma above, there is $c < 1$ such that $f(x^*) = |x^*(x_0)| \leq c$, $\forall x^* \in K$.

Define

$$\eta(\varepsilon, x_0) := 1 - c > 0.$$

Let $x^* \in S_{X^*}$ be such that

$$|x^*(x_0)| > 1 - \eta(\varepsilon, x_0) = c.$$

Since every point $u^* \in K$ satisfies $|u^*(x_0)| \leq c$, the functional x^* above does <u>not</u> belong to K. As $x^* \in B_{X^*}$, it follows that $x^* \in V$, i.e., $x^* \in V \cap S_{X^*}$ and by the choice of V we get $\operatorname{dist}(x^*, M(x_0)) < \varepsilon$. Therefore, there is $y^* \in M(x_0)$ such that $\|y^* - x^*\| < \varepsilon$. As $y^* \in M(x_0)$, $\|y^*\| = |y^*(x_0)| = 1$ and $(X, \mathbb{K})$ has the Lpip, i.e., X is SSD. $\blacksquare$

Prop 2: Let X, Y be finite dim Bs. Then, (X,Y) has the Lp.i.p. In particular, if $Y = \mathbb{K}$ and X is finite-dim, X is SSD.

Proof: Fix $x_0 \in S_X$ and $\varepsilon > 0$. We find $\eta(\varepsilon, x_0) > 0$ such that the definition of Lp.i.p holds true.

Set, once again,

$$M(x_0) := \{ S \in S_{\mathcal{L}(X,Y)} : \| S x_0 \| = 1 \}.$$

This set is non-empty: choose $x_0^* \in S_{X^*}$ with $x_0^*(x_0) = 1$ by H-B; choose $y_0 \in S_Y$ and define $S_0 x := x_0^*(x) y_0$. Then, $S_0 \in M(x_0)$.

Since $S_{\mathcal{L}(X,Y)}$ is compact, $M(x_0)$ is also compact.

Define the open ε-ngh of $M(x_0)$

$$\mathcal{U} := \{ T \in S_{\mathcal{L}(X,Y)} : \operatorname{dist}(T, M(x_0)) < \varepsilon \}.$$

Set $K := S_{\mathcal{L}(X,Y)} \setminus \mathcal{U}$. Then, K is compact.

If $K = \emptyset$, then every operator $T \in S_{\mathcal{L}(X,Y)}$ belongs to $\mathcal{U}$ and therefore there is $S \in M(x_0)$ such that $\| S - T \| < \varepsilon$, and we're done.

Assume then that $K \neq \emptyset$ and define the function $f: K \to \mathbb{R}$ by $f(T) := \|Tx_0\|$. This function is continuous. Notice that, $\forall T \in K$, we have $f(T) < 1$. Indeed, if $T \in S_{\mathcal{L}(X,Y)}$ and $\|Tx_0\| = 1$, then $T \in M(x_0)$. Since $M \subseteq \mathcal{U}$, such an operator does not belong to $K = S_{\mathcal{L}(X,Y)} \setminus \mathcal{U}$. Thus all operators in K satisfy $f(T) < 1$. By the Lemma, there is $c < 1$ such that $\|Tx_0\| \leq c$, $\forall T \in K$. Define once again

$$\eta(\varepsilon, x_0) := 1 - c > 0.$$

Let $T \in S_{\mathcal{L}(X,Y)}$ be such that

$$\|Tx_0\| > 1 - \eta(\varepsilon, x_0) = c.$$

Since every operator in K satisfies $\|Ax_0\| \leq c$, the operator $T \notin K$, i.e., $T \in \mathcal{U}$. By the definition of $\mathcal{U}$, there is $S \in M(x_0)$ such that $\|S - T\| < \varepsilon$, i.e., $\|S\| = \|Sx_0\| = 1$. This shows that (X,Y) has the Lp.p. $\blacksquare$