Let X be a real Banach space. If f is a convex function on a convex subset C of a Banach space and $x \in C$, we define the subdifferential of f at x by

$$\partial f(x) := \{x^* \in X^* : x^*(y-x) \leq f(y) - f(x), \forall y \in C\}.$$

Equivalently, $x^* \in \partial f(x)$ precisely when the affine function $y \mapsto f(x) + x^*(y-x)$ supports the graph of f from below at x [Definition 1.9, Phelps, Convex Functions, Monotone Operators and Differentiability]. With this definition in mind, if $p : X \to \mathbb{R}$, $p(u) := \|u\|$, then

$$\partial p(0) = \{x^* \in X^* : x^*(u) \leq \|u\|, \forall u \in X\}$$

$$= B_{X^*}.$$

Also, if $x \neq 0$, then this definition also gives

$$x^* \in \partial \|\cdot\|(x) \iff x^*(u-x) \leq \|u\| - \|x\|, \forall u \in X$$

and

$$\partial \|\cdot\|(x) = \{x^* \in S_{X^*} : x^*(x) = \|x\|\}. \quad (\#_0)$$

We also need more notation.

We will denote here

$$\Pi_{\pm}(X) := \{(y,y^*) \in S_X \times S_{X^*} : |y^*(y)| = 1\}$$

and, for $0 < \eta < 2$,

$$\Pi^{s}_{\eta}(X,\mathbb{R}) := \{(x,x^*) \in S_X \times S_{X^*} : |x^*(x)| > 1-\eta\}$$

and

$$\Phi^{s}(X,\mathbb{R},\eta) := \sup_{(x,x^*)\in \Pi^{s}_{\eta}(X,\mathbb{R})} \ \inf_{(y,y^*)\in \Pi_{\pm}(X)} \max\{\|x-y\|, \|x^*-y^*\|\}.$$

<u>Fact</u>: Let X be a real Banach space and let $0 < \eta < 2$. Then,

$$\Phi^{s}(X,\mathbb{R},\eta) \leq \min\{\sqrt{2\eta}, 1\}.$$

(I think the estimate by 1 is independent of $\eta > 0$).

Lemma 1: Let $\varphi : X \to \mathbb{R}$ be continuous, convex and bounded below. For every $\alpha > 0$, there are $z \in X$ and $e \in \partial\varphi(z)$ such that $\|e\| \leq \alpha$.

Proof: Choose some $u_0 \in X$ with

$$\varphi(u_0) < \inf_X \varphi + \alpha^2.$$

By Ekeland's [Lemma 3.13, Phelps], there is $z \in X$ such that the function $u \mapsto \varphi(u) + \alpha \cdot \|u - z\|$ has a minimum at z. Then, $0 \in \partial\varphi(z) + \alpha B_{X^*}$. This means that there is $e \in \partial\varphi(z)$ with $\|e\| \leq \alpha$. ∎

Lemma 2: Let X be a real Banach space. For every $(x, f) \in S_X \times S_{X^*}$, we have that

$$\inf_{(y,F) \in \Pi_{\pm}(X)} \max\{\|x - y\|, \|f - F\|\} \leq 1.$$

Proof: Fix $(x, f) \in S_X \times S_{X^*}$. Let $0 < \alpha < 1/2$. Consider $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(u) := \|u\| + \|x - u\| - f(u) \qquad (u \in X).$$

Since $f \in X^*$, φ is convex and also continuous. Notice that φ is also bounded below as

$$\|u\| + \|x-u\| - f(u) \geq \|u\| + (\|u\| - 1) - \|u\|$$

$$= \|u\| - 1.$$

By Lemma 1, there are $z \in X$ and $e \in \partial\varphi(z)$ with $\|e\| \leq \alpha$. Let us write $\boxed{v = x - z.}$ By the subdifferential sum rule [Theorem 3.16, Phelps' book], applied to the three continuous convex summands

$$u \mapsto \|u\| \,, \quad u \mapsto \|x-u\| \quad \text{and} \quad u \mapsto -f(u)$$

there are $g \in \partial\|\cdot\|(z)$ and $h \in \partial\|\cdot\|(v)$ such that

$$e = g - h - f \qquad (\#_1)$$

(notice that the minus in front of h comes from the change of variables in $u \mapsto \|x-u\|$; indeed, the subdifferential at z of this function is $-\partial\|\cdot\|(x-z)$ $= -\partial\|\cdot\|(v)$).

If $z = 0$, then $v = x$ and $h \in \partial\|\cdot\|(x)$. Recalling the notation $(\#_0)$, this means that $h \in S_{X^*}$ and $h(x) = 1$. Taking $y = x$ and $F = -h$, we have

$(y,F) \in \Pi_{\pm}(X)$ and by $(\#_1)$

$$\|f - F\| = \|f + h\| = \|g - e\| \leq 1+\alpha$$

(as $g \in \partial\|\cdot\|(0) = B_{X^*}$) and $\|x - y\| = 0$.

If $v = 0$, then $x = z$ and $g \in \partial\|\cdot\|(x)$. Take $y = x$ and $F = g$. So, $(y,F) \in \Pi_{\pm}(X)$ and

$$\|f - F\| = \|f - g\| = \|-e - f\| \leq 1+\alpha$$

(since $h \in \partial\|\cdot\|(0) = B_{X^*}$) and $\|x - y\| = 0$.

Assume that $z \neq 0$ and $v \neq 0$. Again by the notation $(\#_0)$, we have $g, h \in S_{X^*}$ with $g(z) = \|z\|$ and $h(v) = \|v\|$. Also, by $(\#_1)$,

$$\|g - h\| = \|e + f\| \leq 1+\alpha. \qquad (\#_2)$$

Let us estimate $\|z\|$ and $\|v\|$. Since $x = z + v$ and $\|x\| = 1$, we have

$$g(v) = g(x) - g(z) \leq 1 - \|z\|.$$

On the other hand, using $(\#_2)$ above

$$\|v\| - g(v) = h(v) - g(v) \leq \|h - g\|\cdot\|v\| \leq (1+\alpha)\|v\|.$$

Thus, we have $g(v) \geq \|v\| - (1+\alpha)\|v\| = -\alpha\|v\|$.

So,

$$\|z\| \leq 1 - g(v) \leq 1 + \alpha\|v\| . \quad (\#_3)$$

By symmetry (applying the same argument with h, z instead of g, v), one gets

$$\|v\| \leq 1 + \alpha\|z\| . \quad (\#_4)$$

Combining $(\#_3)$ and $(\#_4)$, we get

$$\|z\| \leq 1 + \alpha\|v\| \leq 1 + \alpha(1 + \alpha\|z\|)$$
$$= 1 + \alpha + \alpha^2\|z\|$$

$$\Rightarrow \|z\| - \alpha^2\|z\| \leq 1 + \alpha$$

$$\Rightarrow (1 - \alpha^2)\|z\| \leq 1 + \alpha$$

$$\Rightarrow \|z\| \leq \frac{1}{1-\alpha} .$$

The same inequality holds for $\|v\|$. So,

$$\|z\| \leq \frac{1}{1-\alpha} \quad \text{and} \quad \|v\| \leq \frac{1}{1-\alpha} . \quad (\#_5)$$

If $\|z\| \geqslant \|v\|$, then we take $y = \dfrac{z}{\|z\|}$ and $F = g$. Then, $(y, F) \in \Pi_1(X)$ - in fact, $F(y) = 1$ in this case. Again using $(\#_1)$, we obtain

$$\|f - F\| = \|f - g\| = \|-h - e\| \leqslant 1 + \alpha$$

and

$$\|x - y\| = \left\| v + z - \frac{z}{\|z\|} \right\| \leqslant \|v\| + |\|z\| - 1|.$$

If $\|z\| \leqslant 1$, then

$$\|v\| + |\|z\| - 1| = \|v\| + 1 - \|z\| \leqslant 1$$

as $\|z\| \geqslant \|v\|$. If $\|z\| > 1$, then we use the assumption $\|z\| \geqslant \|v\|$ and $(\#_5)$ and get

$$\begin{aligned} \|v\| + |\|z\| - 1| &= \|v\| + \|z\| - 1 \\ &\leqslant 2\|z\| - 1 \\ &\overset{(\#_5)}{\leqslant} \frac{2}{1-\alpha} - 1 \\ &= \frac{2 - 1 + \alpha}{1 - \alpha} = \frac{1+\alpha}{1-\alpha}. \end{aligned}$$

If $\|v\| \geq \|z\|$, then we take $y = \dfrac{v}{\|v\|}$ and $F = -h$.

Then, $(y, F) \in \Pi_{\pm}(X)$ as $F(y) = -1$. Again by $(\#_1)$, we have

$$\|f - F\| = \|f + h\| = \|g - e\| \leq 1 + \alpha$$

and the same estimate as above, again by symmetry, gives

$$\|x - y\| \leq \frac{1+\alpha}{1-\alpha}.$$

Consequently, $\forall\, 0 < \alpha < 1/2$, we have that

$$\inf_{(y,F) \in \Pi_{\pm}(X)} \max\{\|x - y\|, \|f - F\|\} \leq \frac{1+\alpha}{1-\alpha}.$$

We now let α goes to 0 and we're done. $\blacksquare$

Now we are ready to prove <u>Fact</u>.

Proof of the Fact: By Lemma 2, we have that $\Phi^S(X,\mathbb{R},\eta) \leq 1$, $\forall \eta > 0$ as $\Pi^S_\eta(X,\mathbb{R}) \subseteq S_X \times S_{X^*}$.

On the other hand, let $0 < \eta < 2$ and take $(x,f) \in \Pi^S_\eta(X,\mathbb{R})$. Pick $\theta \in \{-1,1\}$ to be such that $(\theta f)(x) = |f(x)| > 1-\eta$. By [Corollary 2.4, Chica, Kadets, Martín, Moreno-Pulido, Rambla-Barreno], there is $(y,G) \in S_X \times S_{X^*}$ such that $G(y) = 1$, $\|x-y\| < \sqrt{2\eta}$ and $\|\theta f - G\| < \sqrt{2\eta}$. Now, $(y, \theta G) \in \Pi_{\pm}(X)$ and $\|f - \theta G\| = \|\theta f - G\| < \sqrt{2\eta}$. So,

$$\inf_{(u,H)\in \Pi_{\pm}(X)} \max\{\|x-u\|, \|f-H\|\} \leq \sqrt{2\eta}.$$

Taking supremum over all $(x,f) \in \Pi^S_\eta(X,\mathbb{R})$ gives us $\Phi^S(X,\mathbb{R},\eta) \leq \sqrt{2\eta}$. This shows the result. $\blacksquare$