1 Lecture #1: Week 1

1.1 The real line

Calculus begins with real numbers. This may sound too elementary, but almost every definition that we will use later involves the order of the real line, the distance between two numbers or the possibility of approaching one number by means of others. It is therefore useful to make our notation precise from the beginning.

We use the familiar inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Here $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$ is the set of integers and $\mathbb{Q}$ is the set of rational numbers. A rational number can be written as p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The numbers in $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational*. For instance, $\sqrt{2}$, π and e are irrational.

The real numbers are represented by the points of an oriented line. If $a < b$, then a lies to the left of b. The order is compatible with the arithmetic operations, but one has to remember what happens when multiplying an inequality by a negative number. For instance,

$$-2x < 6 \quad \Longleftrightarrow \quad x > -3.$$

The direction changes because we have multiplied by -1/2 (or, which is the same, divided by -2).

Definition 1.1. Let $A \subseteq \mathbb{R}$.

- The set A is **bounded above** if there is $M \in \mathbb{R}$ such that $x \leqslant M$ for every $x \in A$. Such a number M is called an upper bound.
- The set A is **bounded below** if there is $m \in \mathbb{R}$ such that $m \leqslant x$ for every $x \in A$. Such a number m is called a lower bound.
- The set is **bounded** if it is bounded above and below.

If a non-empty set is bounded above, the least of its upper bounds is called its *supremum* and is denoted by $\sup A$. Similarly, the greatest lower bound is its *infimum*, denoted by $\inf A$. These numbers need not belong to the set.

Example 1.2. Consider $A = (0, 1)$. The number 1 is an upper bound and no smaller number is an upper bound. Consequently, $\sup A = 1$. Nevertheless, $1 \notin A$, so A has no maximum. In the same way, $\inf A = 0$, but A has no minimum. If $B = (0, 1]$, then $\sup B = 1$ and, this time, $\max B = 1$.

The fact that every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum is called the **completeness property** of the real numbers. This property is one of the essential differences between $\mathbb{R}$ and $\mathbb{Q}$. We will not use it explicitly in every calculation, but it is behind the convergence theorems that appear later.

1.2 Intervals

Intervals are the natural domains for functions of one real variable. Let $a < b$. We use the notation

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\},$$
$$(a,b] = \{x \in \mathbb{R} : a < x \leqslant b\}, \qquad [a,b) = \{x \in \mathbb{R} : a \leqslant x < b\}.$$

Intervals involving infinity are always open at the infinite end. Thus,

$$(-\infty, b] = \{x \in \mathbb{R} : x \leqslant b\}, \qquad (a, +\infty) = \{x \in \mathbb{R} : x > a\}.$$

The symbols $+\infty$ and $-\infty$ are not real numbers, so it would make no sense to include them as endpoints.

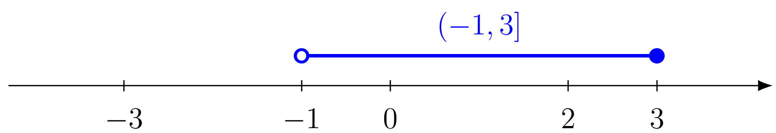

Figure 1: The left endpoint is excluded and the right endpoint is included.

Unions and intersections of intervals occur frequently. The intersection contains the points that belong to both sets, while the union contains the points that belong to at least one of them.

Example 1.3. Let $A = [-2, 4)$ and $B = (1, 6]$. Then

$$A \cap B = (1,4) \qquad \text{and} \qquad A \cup B = [-2, 6].$$

Notice that 1 does not belong to the intersection because $1 \notin B$, and 4 does not belong to the intersection because $4 \notin A$.

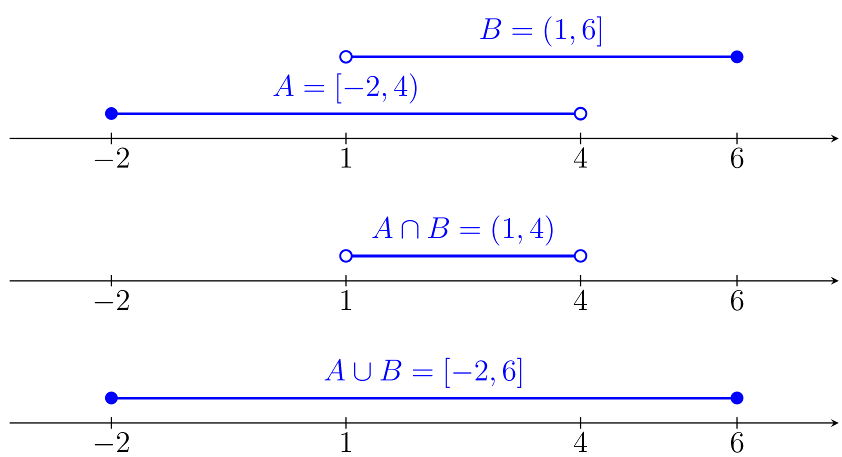

Figure 2: The sets A and B, their intersection and their union.

1.3 Absolute value and distance

Definition 1.4. The **absolute value** of $x \in \mathbb{R}$ is

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The distance between $x, y \in \mathbb{R}$ is $|x-y|$.

Geometrically, $|x|$ is the distance from x to the origin. This interpretation is often much quicker than splitting an expression into cases. For $r > 0$,

$$|x-a| < r \quad \Longleftrightarrow \quad a-r < x < a+r.$$

In words, x is at distance less than r from a. Similarly,

$$|x-a| \leqslant r \quad \Longleftrightarrow \quad x \in [a-r, a+r].$$

Proposition 1.5 (Basic properties of the absolute value)**.** For all $x, y \in \mathbb{R}$, the following statements hold:

(a) $|x| \geqslant 0$, and $|x| = 0$ if and only if $x = 0$;

(b) $|xy| = |x|\,|y|$;

(c) $|x+y| \leqslant |x| + |y|$;

(d) $\big||x| - |y|\big| \leqslant |x-y|$.

Property (c) is called the **triangle inequality**.

The name triangle inequality comes from geometry: the length of one side of a triangle cannot exceed the sum of the lengths of the other two sides. On the real line it is also easy to understand: travelling from 0 to x and then from x to $x+y$ cannot be shorter than travelling directly from 0 to $x+y$.

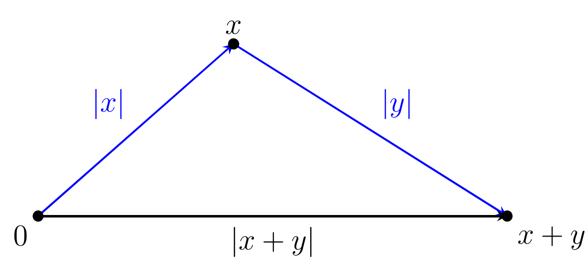

Figure 3: Geometric interpretation of the triangle inequality.

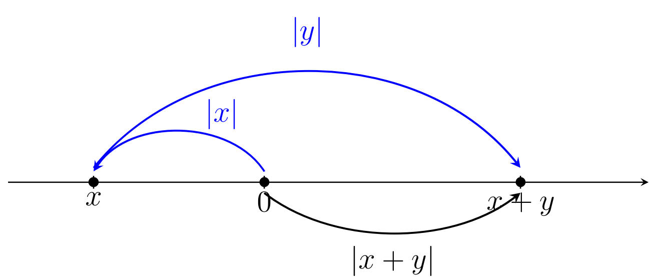

Figure 4: Travelling directly from 0 to $x+y$ is no longer than travelling first from 0 to x and then from x to $x+y$.

Example 1.6. Solve the inequality $|2x - 3| < 5$. We first isolate the expression inside the absolute value:

$$|2x - 3| < 5 \iff -5 < 2x - 3 < 5.$$

Adding 3 and then dividing by 2, we obtain

$$-1 < x < 4.$$

Therefore, the solution set is $(-1, 4)$.

Example 1.7. Solve $|x - 1| \geqslant 3$. Here we are looking for the points whose distance from 1 is at least 3. Thus, x must lie to the left of -2 or to the right of 4:

$$|x - 1| \geqslant 3 \iff x \leqslant -2 \text{ or } x \geqslant 4.$$

The solution is $(-\infty, -2] \cup [4, +\infty)$.

1.4 A useful checklist for inequalities

When solving an inequality, keep the following points in mind.

(1) Determine where every expression is defined. A denominator cannot be zero and the argument of a logarithm must be positive.

(2) When multiplying or dividing by a quantity of unknown sign, split the problem into cases.

(3) Multiplying or dividing by a negative quantity reverses the inequality.

(4) At the end, put together all the information you have collected.

Example 1.8. Let us solve

$$\frac{x - 1}{x + 2} \geqslant 0.$$

The critical points are $x = 1$, where the numerator vanishes, and $x = -2$, where the expression is not defined. They divide the real line into three intervals. Testing one point in each interval gives

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$x - 1$	$-$	$-$	$-$	0	$+$
$x + 2$	$-$	0	$+$	$+$	$+$
$(x - 1)/(x + 2)$	$+$	not defined	$-$	0	$+$

We include $x = 1$ because equality is allowed, but exclude $x = -2$ because it is not in the domain. The solution is

$$(-\infty, -2) \cup [1, +\infty).$$

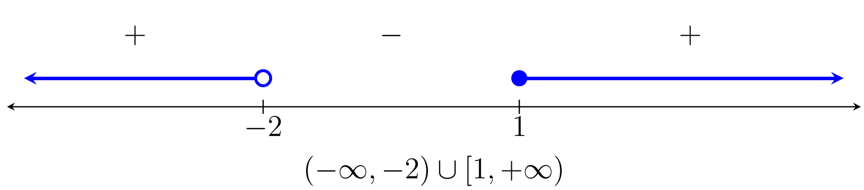

Figure 5: The solution of $\frac{x-1}{x+2} \geqslant 0$.