2 Lecture #2: Week 1

2.1 Functions of one real variable

A function describes a rule that assigns one output to each admissible input. In applications, the input may be time and the output may be position, temperature, voltage, cost or another quantity of interest.

Definition 2.1. Let $A, B \subseteq \mathbb{R}$. A **function** $f : A \to B$ is a rule that assigns to every $x \in A$ exactly one number $f(x) \in B$. The set A is the **domain**, B is the **codomain**, and

$$f(A) = \{f(x) : x \in A\}$$

is the **range** or **image** of f.

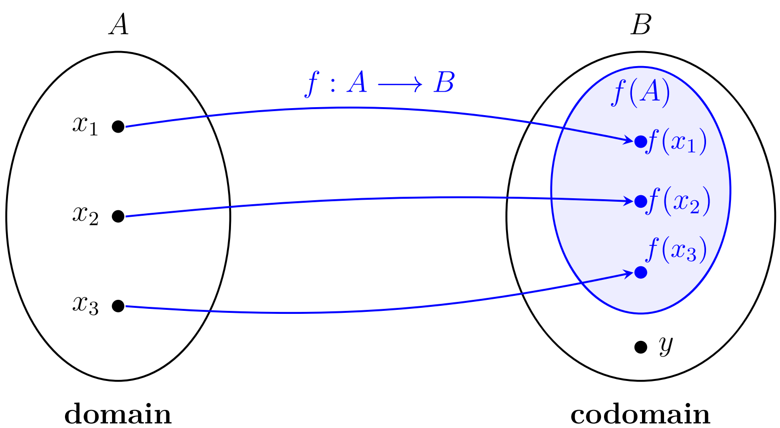

Figure 6: Every point of the domain A has exactly one image in B. The set of values reached by the function is the range $f(A) \subseteq B$.

The phrase "exactly one" is important. A vertical line can meet the graph of a function in at most one point. This is known as the vertical line test.

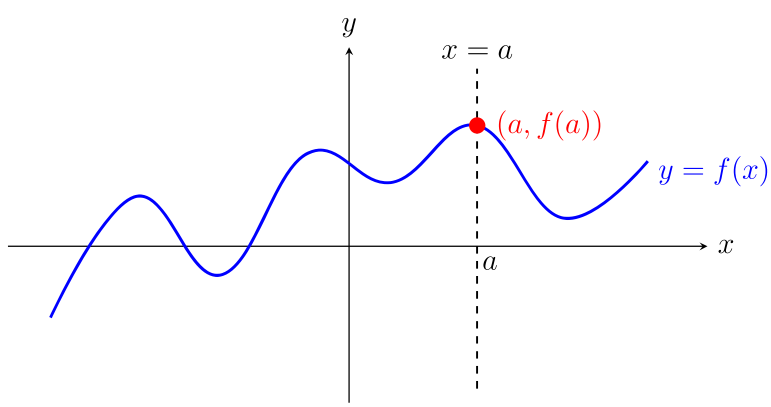

Figure 7: The vertical line $x = a$ meets the graph at exactly one point.

Example 2.2. For $f(x) = \sqrt{4 - x^2}$, the condition $4 - x^2 \geqslant 0$ gives $-2 \leqslant x \leqslant 2$. Thus, the natural domain is $[-2, 2]$. Since $0 \leqslant f(x) \leqslant 2$, the range is $[0, 2]$. Its graph is the upper semicircle of radius 2 centered at the origin.

When a function is given by a formula, its natural domain consists of all real numbers for which the formula makes sense. The most common restrictions are:

- a denominator must be different from zero;
- an even root must have a non-negative radicand;
- a logarithm must have a positive argument.

Example 2.3. Determine the domain of

$$f(x) = \frac{\sqrt{x+1}}{\ln(3-x)}.$$

The square root requires $x \geqslant -1$. The logarithm requires $3 - x > 0$, so $x < 3$. Moreover, the denominator cannot vanish, hence $\ln(3 - x) \neq 0$, which is equivalent to $3 - x \neq 1$, or $x \neq 2$. Therefore,

$$\mathrm{Dom}(f) = [-1, 2) \cup (2, 3).$$

2.2 Elementary functions

The elementary functions used throughout the course are obtained from the following families by sums, products, quotients and compositions:

- polynomial and rational functions;
- power and root functions;
- exponential and logarithmic functions;
- trigonometric and inverse trigonometric functions.

It is useful to remember their domains, ranges and qualitative shapes. For example, e^x is positive and strictly increasing on $\mathbb{R}$, while $\ln x$ is defined only for $x > 0$ and is the inverse of e^x. The functions $\sin x$ and $\cos x$ are bounded between -1 and 1 and have period 2π.

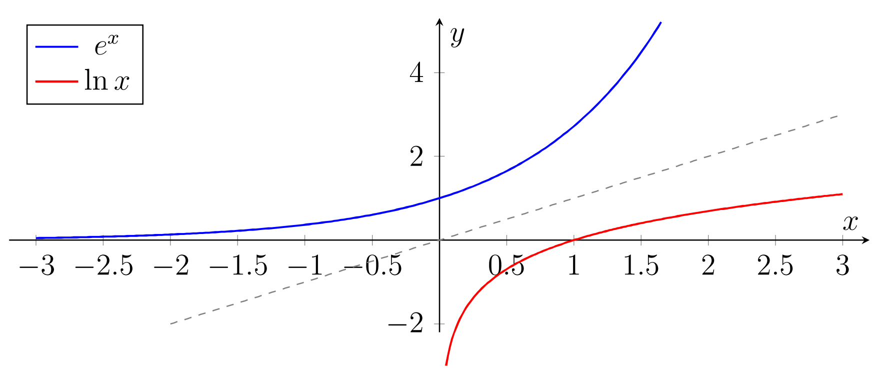

Figure 8: The graphs of e^x and $\ln x$ are symmetric with respect to $y = x$.

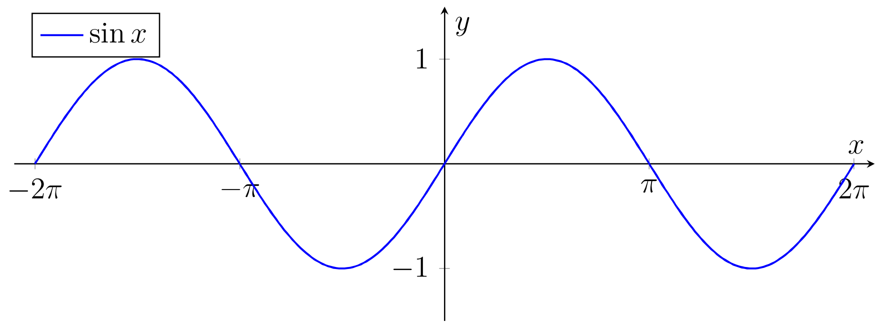

Figure 9: The graph of the sine function $y = \sin x$.

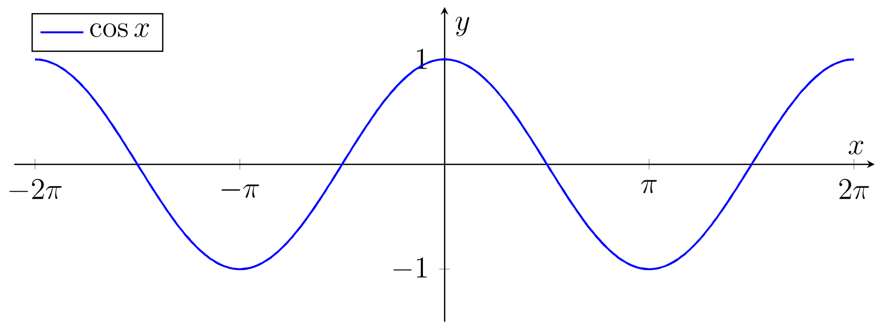

Figure 10: The graph of the cosine function $y = \cos x$.

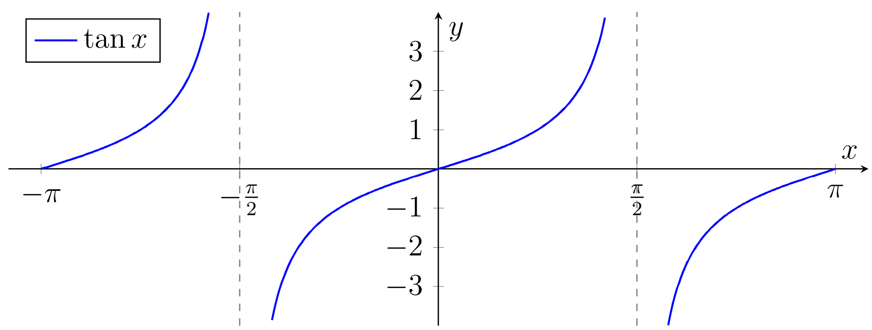

Figure 11: The graph of the tangent function $y = \tan x$. The dashed lines are vertical asymptotes.

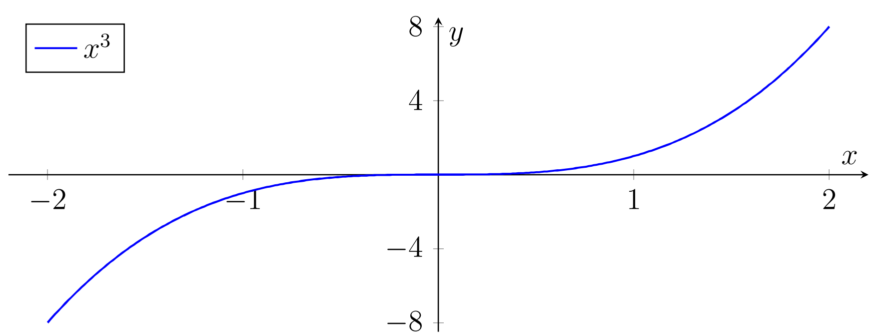

Figure 12: The graph of the cubic function $y = x^3$.

2.3 Composition and inverse functions

If $f : A \to B$ and $g : B \to C$, their composition is defined by

$$(g \circ f)(x) = g(f(x)).$$

The order matters. Indeed, in general, $g \circ f$ and $f \circ g$ are different functions.

Example 2.4. Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Then

$$(g \circ f)(x) = \sqrt{x^2 + 1}, \qquad x \in \mathbb{R},$$

while

$$(f \circ g)(x) = x + 1, \qquad x \geqslant 0.$$

Even when the formulas look simple, the domains must be recorded.

Definition 2.5. A function $f : A \to B$ is **one-to-one** or **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. It is **onto** or **surjective** if $f(A) = B$. If both properties hold, f is **bijective** and has an inverse $f^{-1} : B \to A$.

The graph of an injective function meets every horizontal line at most once. If f is bijective, the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.

Example 2.6. The function $f(x) = x^2$ is not injective on $\mathbb{R}$, because $f(1) = f(-1)$. However, its restriction $f : [0, +\infty) \to [0, +\infty)$ is bijective and

$$f^{-1}(x) = \sqrt{x}.$$

Restricting a domain is often necessary before taking an inverse.

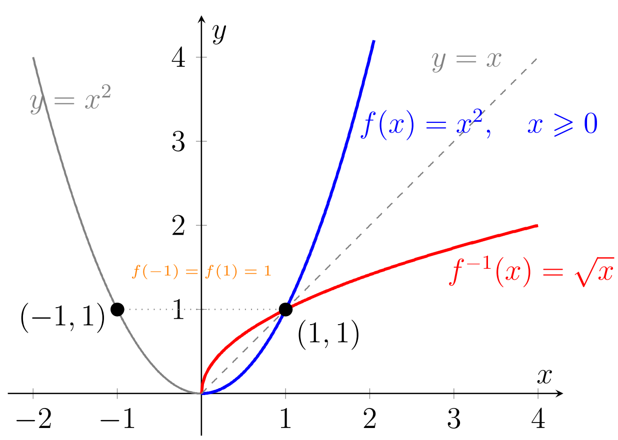

Figure 13: The function $f(x) = x^2$ is not injective on $\mathbb{R}$. After restricting its domain to $[0, +\infty)$, the blue branch is bijective and its inverse is $y = \sqrt{x}$, shown in red. The two graphs are symmetric with respect to the line $y = x$.

2.4 Symmetry and periodicity

Definition 2.7. Let f be a function

- The function is **even** if $f(-x) = f(x)$ for every x in its domain. Its graph is symmetric about the y-axis.
- The function is **odd** if $f(-x) = -f(x)$ for every x in its domain. Its graph is symmetric about the origin.

Definition 2.8. A function $f : D \to \mathbb{R}$ is **periodic** if there is $T > 0$ such that $D + T = D$, where $D + T = \{x + T : x \in D\}$, and $f(x + T) = f(x)$ for every $x \in D$.

For instance, x^2 and $\cos x$ are even; x^3 and $\sin x$ are odd; and $\sin x$ and $\cos x$ are periodic with period 2π. A function need not be even or odd.

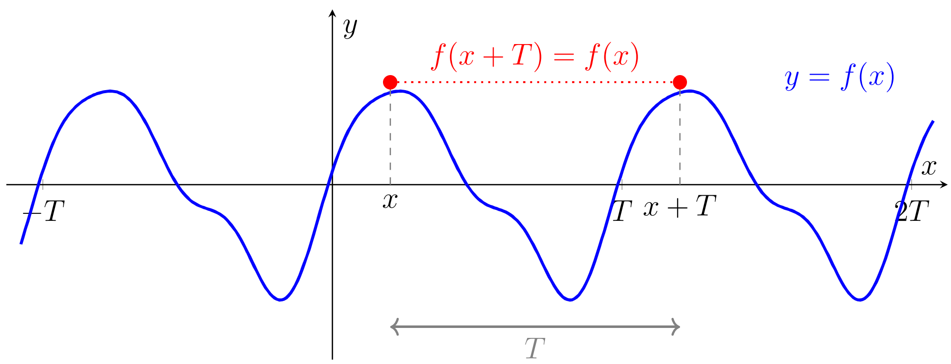

Figure 14: The graph repeats after every interval of length T, so $f(x + T) = f(x)$ for every x.

2.5 Shifting and scaling graphs

Suppose that the graph of $y = f(x)$ is known. The following rules allow us to draw many related graphs without starting from zero. In the scaling rules, we assume $a \neq 0$ and $b \neq 0$.

Function	Effect on the graph of f
$f(x) + k$	shift vertically by k
$f(x - h)$	shift horizontally by h
$af(x)$	scale vertically by the factor $\lvert a\rvert$; reflect if $a < 0$
$f(bx)$	scale horizontally by the factor $1/\lvert b\rvert$; reflect if $b < 0$
$\lvert f(x)\rvert$	reflect the part below the x-axis upward
$f(\lvert x\rvert)$	copy the right half of the graph to the left

The horizontal rule is the one most often remembered incorrectly. In $f(x - h)$, a positive h shifts the graph to the *right*, because the input must be h units larger to produce the same old value.

Example 2.9. Starting from $f(x) = |x|$, consider

$$g(x) = -2|x - 1| + 3.$$

The graph is shifted one unit to the right, stretched vertically by a factor 2, reflected about the x-axis, and shifted three units upward. Its vertex is $(1, 3)$. It meets the x-axis when

$$-2|x-1|+3=0 \quad \iff \quad |x-1|=\frac{3}{2},$$

so the intercepts are $x = -1/2$ and $x = 5/2$.

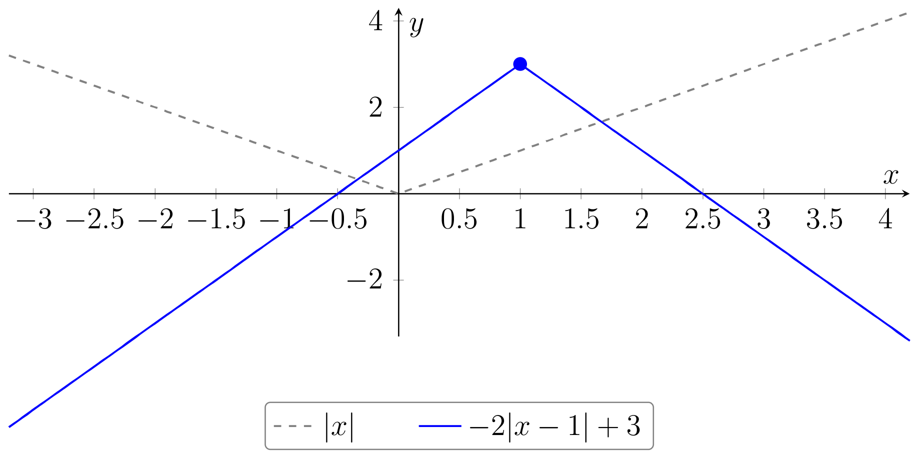

Figure 15: A graph obtained from $y = |x|$ by elementary transformations.