3 Practice #1: Week 1

3.1 The real line and elementary functions

In this class we will solve six problems about intervals, absolute values and domains.

3.2 Problem 1

Solve the inequalities

(a) $|2x-3| \leqslant 5$;

(b) $|x+1| > 2$;

(c) $|x-2| \leqslant |x+1|$.

(a) We write the absolute-value inequality as a double inequality:

$$-5 \leqslant 2x-3 \leqslant 5.$$

Adding 3 and dividing by 2 gives

$$-1 \leqslant x \leqslant 4.$$

Thus, the solution is $[-1,4]$.

(b) The distance from x to -1 must be greater than 2. Therefore,

$$x < -3 \quad \text{or} \quad x > 1,$$

and the solution is $(-\infty,-3) \cup (1,+\infty)$.

(c) Both sides are non-negative, so we may square without changing the inequality:

$$(x-2)^2 \leqslant (x+1)^2.$$

After expanding and cancelling x^2, we obtain

$$-4x+4 \leqslant 2x+1 \quad \Longleftrightarrow \quad x \geqslant \frac{1}{2}.$$

Geometrically, these are precisely the points at least as close to 2 as to -1.

3.3 Problem 2

Let

$$A=[-3,2), \qquad B=(-1,5], \qquad C=\{x \in \mathbb{R} : |x-1| < 3\}.$$

Write C as an interval and calculate $A \cap B$, $A \cup B$, $(A \cap C) \setminus B$, $\sup(A \cap C)$ and $\inf(A \cap C)$.

The inequality defining C gives

$$-3 < x-1 < 3 \quad \Longleftrightarrow \quad -2 < x < 4,$$

so $C = (-2, 4)$. Now
$$A \cap B = (-1, 2), \qquad A \cup B = [-3, 5].$$
Moreover, $A \cap C = (-2, 2)$. Removing the points of $B = (-1, 5]$ leaves
$$(A \cap C) \setminus B = (-2, -1].$$
Finally,
$$\sup(A \cap C) = 2, \qquad \inf(A \cap C) = -2.$$
Neither value belongs to $A \cap C$, so this set has neither a maximum nor a minimum.

3.4 Problem 3

Determine the natural domain of each function:

(a) $f(x) = \dfrac{x+1}{x^2-4}$;

(b) $g(x) = \sqrt{\dfrac{x-1}{x+2}}$;

(c) $h(x) = \ln(4 - x^2)$;

(d) $k(x) = \dfrac{\sqrt{x+3}}{\ln(x-1)}$.

(a) We require $x^2 - 4 \neq 0$, hence
$$\mathrm{Dom}(f) = \mathbb{R} \setminus \{-2, 2\}.$$

(b) We need $(x-1)/(x+2) \geqslant 0$ and $x \neq -2$. The critical points are -2 and 1. A sign chart gives
$$\mathrm{Dom}(g) = (-\infty, -2) \cup [1, +\infty).$$

(c) The logarithm requires $4 - x^2 > 0$, or $x^2 < 4$. Therefore,
$$\mathrm{Dom}(h) = (-2, 2).$$

(d) The square root requires $x \geqslant -3$, the logarithm requires $x > 1$, and the denominator requires $\ln(x-1) \neq 0$, so $x \neq 2$. Combining the conditions,
$$\mathrm{Dom}(k) = (1, 2) \cup (2, +\infty).$$

3.5 Problem 4

Let $f(x) = \sqrt{x-1}$ and $g(x) = x^2 - 4$. Find $f \circ g$ and $g \circ f$, including their natural domains.

We have

$$(f \circ g)(x) = f(x^2 - 4) = \sqrt{x^2 - 5}.$$

The radicand must be non-negative, so $x^2 \geqslant 5$. Hence,

$$\mathrm{Dom}(f \circ g) = (-\infty, -\sqrt{5}] \cup [\sqrt{5}, +\infty).$$

In the opposite order,

$$(g \circ f)(x) = g(\sqrt{x-1}) = (\sqrt{x-1})^2 - 4 = x - 5.$$

Although the final formula $x - 5$ is defined for every real number, the composition still requires $x \geqslant 1$ because $f(x)$ must exist. Thus,

$$\mathrm{Dom}(g \circ f) = [1, +\infty).$$

3.6 Problem 5

Consider

$$f(x) = \frac{3x-2}{x+4}, \qquad x \neq -4.$$

Show that f is one-to-one, determine its range and find its inverse.

Suppose that $f(x_1) = f(x_2)$. Then

$$\frac{3x_1 - 2}{x_1 + 4} = \frac{3x_2 - 2}{x_2 + 4}.$$

Cross-multiplication gives

$$(3x_1 - 2)(x_2 + 4) = (3x_2 - 2)(x_1 + 4).$$

After expanding and cancelling equal terms, $14x_1 = 14x_2$, hence $x_1 = x_2$. Therefore, f is injective.

To find the inverse, put $y = (3x-2)/(x+4)$ and solve for x:

$$yx + 4y = 3x - 2 \quad \Longleftrightarrow \quad x(y-3) = -2 - 4y.$$

Thus,

$$x = \frac{2+4y}{3-y}.$$

This expression is defined when $y \neq 3$, and the corresponding value of x can never be -4: the equation $(2+4y)/(3-y) = -4$ would give $2 = -12$. Conversely, $f(x)$ can never equal 3. Consequently,

$$f(\mathbb{R} \setminus \{-4\}) = \mathbb{R} \setminus \{3\}, \qquad f^{-1}(x) = \frac{2+4x}{3-x}.$$