6 Practice #2: Week 2

6.1 Limits and continuity

In this class we will solve seven problems about finite and one-sided limits, indeterminate forms, continuity, existence theorems and rates of change.

6.2 Problem 1

Calculate the following limits:

(a) $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$;

(b) $\lim_{x\to 4} \dfrac{\sqrt{x+5}-3}{x-4}$;

(c) $\lim_{x\to 2} \dfrac{1/x-1/2}{x-2}$.

(a) Direct substitution gives $0/0$, so we factor before taking the limit:

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3 \qquad (x \neq 3).$$

Therefore, the limit is 6.

(b) We rationalize the numerator:

$$\begin{aligned} \frac{\sqrt{x+5}-3}{x-4} &= \frac{\sqrt{x+5}-3}{x-4}\,\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3} \\ &= \frac{x+5-9}{(x-4)(\sqrt{x+5}+3)} = \frac{1}{\sqrt{x+5}+3}. \end{aligned}$$

Hence, the limit is $1/6$.

(c) Combining the fractions in the numerator gives

$$\frac{1/x-1/2}{x-2} = \frac{(2-x)/(2x)}{x-2} = -\frac{1}{2x}, \qquad x \neq 2.$$

Thus, the limit is $-1/4$.

6.3 Problem 2

Use standard limits to calculate

(a) $\lim_{x\to 0} \dfrac{\sin(4x)}{x}$;

(b) $\lim_{x\to 0} \dfrac{1-\cos(3x)}{x^2}$;

(c) $\lim_{x\to 0} \dfrac{e^{5x}-1}{\ln(1+2x)}$;

(d) $\lim_{x\to 0} (1+3x)^{2/x}$.

(a) We write

$$\frac{\sin(4x)}{x} = 4\frac{\sin(4x)}{4x}.$$

The quotient on the right tends to 1, so the answer is 4.

(b) Put $u = 3x$. Then

$$\frac{1-\cos(3x)}{x^2} = 9\frac{1-\cos u}{u^2} \longrightarrow \frac{9}{2}.$$

(c) We separate the two standard limits:

$$\frac{e^{5x}-1}{\ln(1+2x)} = \frac{e^{5x}-1}{5x} \cdot \frac{2x}{\ln(1+2x)} \cdot \frac{5}{2}.$$

The first two factors tend to 1, and therefore the limit is $5/2$.

(d) We have an indeterminate form 1^∞. Since

$$(1+3x)^{2/x} = \left((1+3x)^{1/(3x)}\right)^6,$$

the limit is e^6.

6.4 Problem 3

Study all one-sided limits at the indicated point and decide whether the two-sided limit exists:

(a) $f(x) = \dfrac{x+1}{|x+1|}$ at $x = -1$;

(b) $g(x) = \dfrac{1}{x-2}$ at $x = 2$;

(c) $h(x) = \dfrac{|x|}{x}$ at $x = 0$.

(a) If $x < -1$, then $x + 1 < 0$ and $|x+1| = -(x+1)$, so $f(x) = -1$. If $x > -1$, then $f(x) = 1$. Consequently,

$$\lim_{x\to -1^-} f(x) = -1, \qquad \lim_{x\to -1^+} f(x) = 1.$$

The two-sided limit does not exist.

(b) To the left of 2, the denominator is negative and approaches zero; to the right it is positive and approaches zero. Hence,

$$\lim_{x\to 2^-} g(x) = -\infty, \qquad \lim_{x\to 2^+} g(x) = +\infty.$$

Again, there is no two-sided limit.

(c) If $x < 0$, then $|x| = -x$ and $h(x) = -1$. If $x > 0$, then $h(x) = 1$. Thus, the one-sided limits are -1 and 1, and the two-sided limit does not exist.

6.5 Problem 4

Calculate

(a) $\lim\limits_{x\to+\infty} \dfrac{5x^3 - 2x + 1}{2x^3 + x^2}$;

(b) $\lim\limits_{x\to+\infty} (\sqrt{x^2 + 3x} - x)$;

(c) $\lim\limits_{x\to+\infty} x \sin(1/x)$.

(a) Divide numerator and denominator by x^3:

$$\frac{5 - 2/x^2 + 1/x^3}{2 + 1/x} \longrightarrow \frac{5}{2}.$$

(b) Rationalization gives

$$\sqrt{x^2 + 3x} - x = \frac{3}{\sqrt{1 + 3/x} + 1} \longrightarrow \frac{3}{2}.$$

(c) Put $u = 1/x$. Then $u \to 0^+$ and

$$x \sin(1/x) = \frac{\sin u}{u} \longrightarrow 1.$$

6.6 Problem 5

Find $a, b \in \mathbb{R}$ so that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x < 1, \\ ax + b, & 1 \leqslant x < 2, \\ 5, & x \geqslant 2 \end{cases}$$

is continuous on $\mathbb{R}$.

Each formula is continuous inside its own interval, so only $x = 1$ and $x = 2$ require attention. For $x < 1$ and $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = x + 1.$$

Thus, continuity at 1 requires

$$a + b = f(1) = \lim_{x\to1^-} f(x) = 2.$$

At $x = 2$, the right-hand value and right-hand limit are 5, while the left-hand limit is $2a + b$. Hence,

$$2a + b = 5.$$

Subtracting the equations gives $a = 3$, and then $b = -1$. These values satisfy both junction conditions, so they make f continuous on all of $\mathbb{R}$.

6.7 Problem 6

Classify the discontinuity in each case and repair it if this is possible by changing only the value at the indicated point.

(a) $f(x) = (x^2 - 4)/(x - 2)$ for $x \neq 2$, with $f(2) = 7$;

(b) $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x \geqslant 0$;

(c) $h(x) = 1/(x - 1)^2$ at $x = 1$.

(a) For $x \neq 2$, $f(x) = x + 2$, so

$$\lim_{x\to 2} f(x) = 4 \neq f(2).$$

This is removable. Replacing $f(2)$ by 4 makes the function continuous.

(b) We have

$$\lim_{x\to 0^-} g(x) = 0, \qquad \lim_{x\to 0^+} g(x) = 1.$$

This is a jump discontinuity. Changing the single value $g(0)$ cannot make the two one-sided limits agree, so it cannot be repaired in this way.

(c) Both one-sided limits are $+\infty$. This is an infinite discontinuity, and assigning a finite value to $h(1)$ cannot repair it.

6.8 Problem 7

(a) Prove that every continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.

(b) If the position of a particle is $s(t) = t^2 - 3t$, calculate its average velocity from $t = 1$ to $t = 1 + h$ and its instantaneous velocity at $t = 1$.

(a) Define $F(x) = f(x) - x$. This function is continuous on $[0, 1]$. Since f takes values in $[0, 1]$,

$$F(0) = f(0) \geqslant 0, \qquad F(1) = f(1) - 1 \leqslant 0.$$

If either endpoint value is zero, that endpoint is already a fixed point. Otherwise, $F(0) > 0 > F(1)$, and Bolzano's theorem gives $c \in (0, 1)$ with $F(c) = 0$. Thus, $f(c) = c$.

(b) We compute

$$\begin{aligned}\frac{s(1+h) - s(1)}{h} &= \frac{(1+h)^2 - 3(1+h) - (-2)}{h} \\ &= \frac{-h + h^2}{h} = -1 + h.\end{aligned}$$

Therefore, the instantaneous velocity is

$$\lim_{h \to 0}(-1 + h) = -1.$$

The negative sign means that, at $t = 1$, the position is decreasing.