4 Lecture #3: Week 2

4.1 The idea of a limit

In the previous week we described functions by means of formulas and graphs. We now want to understand what a function does *close to* a given point. This is not always the same as asking for the value of the function at that point. For instance, consider

$$f(x) = \frac{x^2 - 1}{x - 1}, \qquad x \neq 1.$$

The function is not defined at $x = 1$. Nevertheless, whenever $x \neq 1$, we may factor the numerator and obtain

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1.$$

Thus, when x is close to 1, the value $f(x)$ is close to 2. We express this by writing

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Notice that the calculation never required a value for $f(1)$. A limit describes the behaviour around the point, not necessarily at the point itself.

Definition 4.1. Let $f : A \to \mathbb{R}$, let a be a point that can be approached by points of A different from a and let $L \in \mathbb{R}$. We say that

$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in A, \quad 0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

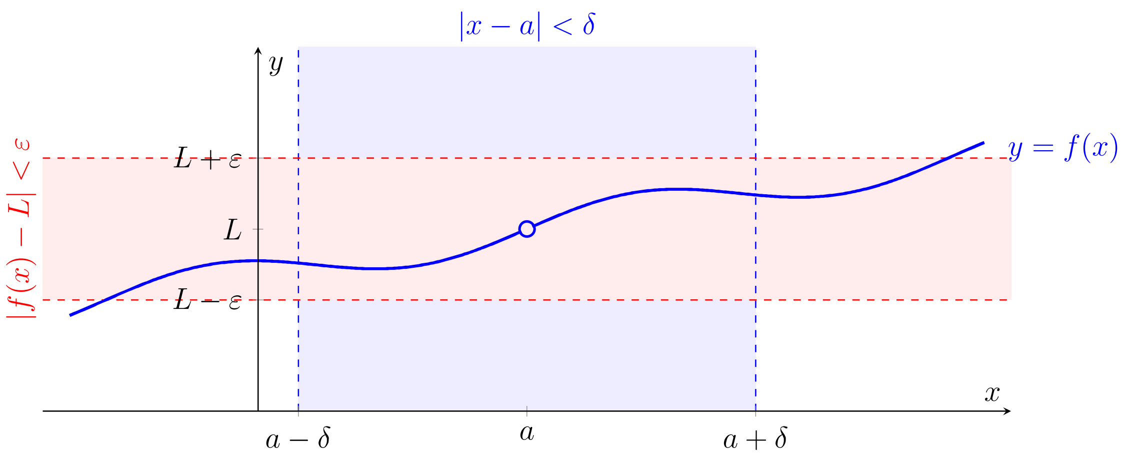

Figure 16: Whenever $0 < |x - a| < \delta$, the value $f(x)$ lies between $L - \varepsilon$ and $L + \varepsilon$. The value at $x = a$ is irrelevant.

The number ε measures how close we want $f(x)$ to be to L, while δ measures how close x must be to a. The puncture $0 < |x - a|$ is essential: it tells us that the value at a is ignored.

Example 4.2 (Reading the ε–δ definition)**.** Let us verify directly that

$$\lim_{x\to 3}(2x-1)=5.$$

Given $\varepsilon>0$, we want $|(2x-1)-5|<\varepsilon$. But

$$|(2x-1)-5|=|2x-6|=2|x-3|.$$

It is therefore enough to require $|x-3|<\varepsilon/2$. Taking $\delta=\varepsilon/2$, we obtain

$$0<|x-3|<\delta \quad\Longrightarrow\quad |(2x-1)-5|<2\delta=\varepsilon.$$

This proves the limit. In more complicated examples, finding a suitable δ may require additional estimates, but the logical structure is always the same: the desired accuracy in the output determines a sufficient accuracy in the input.

There is another useful way to recognize a limit. If every sequence of points $x_n\neq a$ approaching a has images $f(x_n)$ approaching the same number L, then the limit is L. Conversely, if we can find two ways of approaching a that produce different limiting values, the limit does not exist. For example, $\sin(1/x)$ has no limit at 0. Along

$$x_n=\frac{1}{\pi/2+2\pi n}$$

the function is always equal to 1, whereas along

$$y_n=\frac{1}{3\pi/2+2\pi n}$$

it is always equal to -1. Both sequences approach 0, but their images have different limits. This gives a precise reason for the oscillation seen in the graph.

Example 4.3. Let

$$f(x)=\begin{cases}3x-1, & x\neq 2,\\ 100, & x=2.\end{cases}$$

The exceptional value $f(2)=100$ has no effect on the limit. Since $3x-1$ approaches 5 as x approaches 2, we have

$$\lim_{x\to 2}f(x)=5,$$

even though $f(2)\neq 5$.

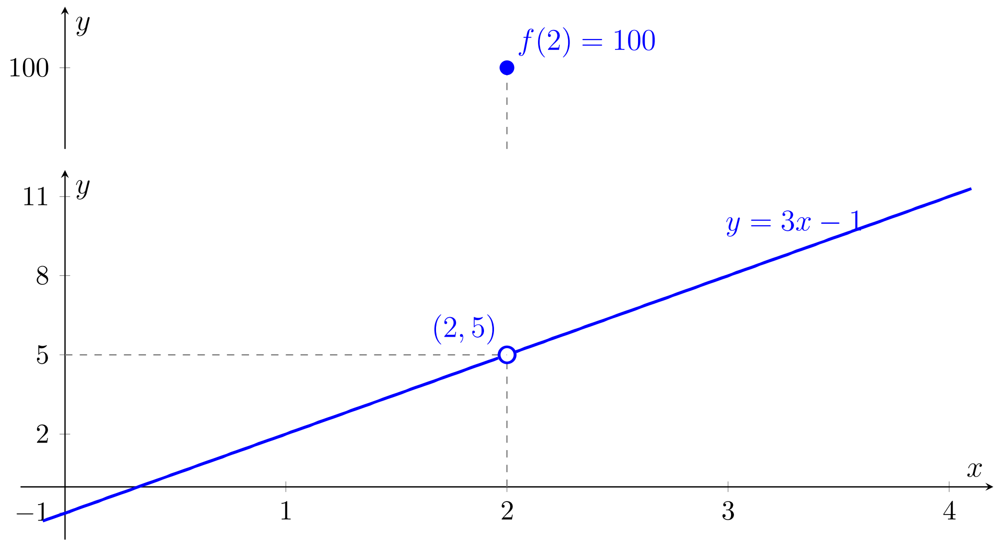

Figure 17: The graph approaches the open point $(2, 5)$, while the actual value $f(2) = 100$ is shown separately. Hence $\lim_{x\to 2} f(x) = 5 \neq f(2)$.

4.2 Algebra of finite limits

Limits behave well with the ordinary arithmetic operations. This is the reason why many limits can be calculated simply by substitution.

Theorem 4.4 (Limit laws)**.** Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

$$\begin{aligned} \lim_{x\to a}(f(x) + g(x)) &= L + M, \\ \lim_{x\to a} f(x)g(x) &= LM, \\ \lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{L}{M}, \qquad \text{provided } M \neq 0. \end{aligned}$$

Moreover, constants may be taken outside limits, and integer powers and roots may be passed through a limit whenever the resulting expressions are defined.

It follows that every polynomial is continuous with respect to taking limits:

$$\lim_{x\to a} P(x) = P(a).$$

The same is true for a rational function P/Q at every point where $Q(a) \neq 0$.

Example 4.5. By direct substitution,

$$\lim_{x\to 2} \frac{x^3 + 4x - 1}{x^2 + 1} = \frac{2^3 + 4 \cdot 2 - 1}{2^2 + 1} = 3.$$

There is no indeterminacy because the denominator approaches 5, not zero.

The proof of the Squeeze theorem could be one of the questions in the oral exam in January.

Theorem 4.6 (Squeeze theorem)**.** Suppose that $g(x) \leqslant f(x) \leqslant h(x)$ for every x sufficiently close to a, with the possible exception of $x = a$. If

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} f(x) = L$.

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to a} g(x) = L$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies $L - \varepsilon < g(x) < L + \varepsilon$. Similarly, since $\lim_{x\to a} h(x) = L$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2$ implies $L - \varepsilon < h(x) < L + \varepsilon$. Moreover, there exists $\delta_0 > 0$ such that $g(x) \leqslant f(x) \leqslant h(x)$ whenever $0 < |x - a| < \delta_0$. Set

$$\delta = \min\{\delta_0, \delta_1, \delta_2\}.$$

If $0 < |x - a| < \delta$, then

$$L - \varepsilon < g(x) \leqslant f(x) \leqslant h(x) < L + \varepsilon.$$

Therefore,

$$|f(x) - L| < \varepsilon.$$

Hence $\lim_{x\to a} f(x) = L$. □

Example 4.7. Since $-1 \leqslant \sin(1/x) \leqslant 1$, multiplication by $|x|$ gives

$$-|x| \leqslant x \sin(1/x) \leqslant |x|.$$

Both outer functions tend to zero as $x \to 0$. Therefore,

$$\lim_{x\to 0} x \sin(1/x) = 0.$$

The factor $\sin(1/x)$ oscillates infinitely often, but it remains bounded and the factor x forces the product to zero.

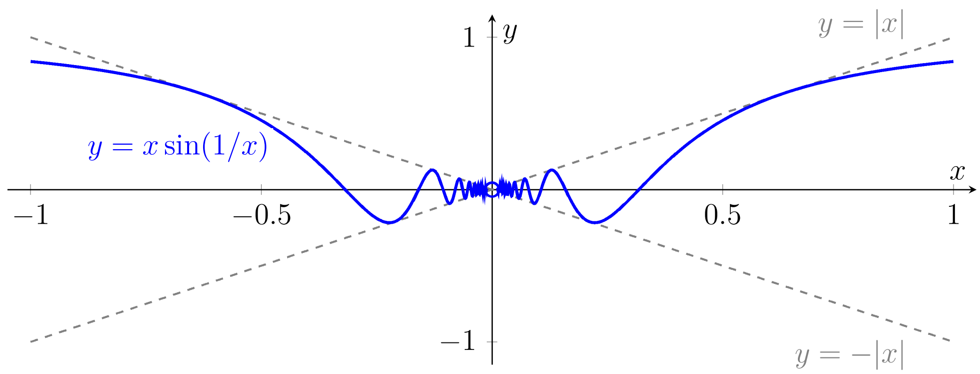

Figure 18: The graph of $x \sin(1/x)$ oscillates infinitely often near zero, but it remains trapped between $-|x|$ and $|x|$.

4.3 One-sided limits

Sometimes the behaviour to the left of a point and to its right is different. We then need one-sided limits.

Definition 4.8. The **left-hand limit** $\lim_{x\to a^-} f(x)$ is obtained by approaching a through values $x < a$. The **right-hand limit** $\lim_{x\to a^+} f(x)$ is obtained through values $x > a$.

The proof of the following result could be also something that it can be in the oral exam.

Proposition 4.9. If both sides of a belong to the domain near a, then

$$\lim_{x\to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Proof. Suppose first that $\lim_{x\to a} f(x) = L$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$. In particular, this holds when $a - \delta < x < a$ and when $a < x < a + \delta$. Therefore,

$$\lim_{x\to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = L.$$

Conversely, suppose that both one-sided limits are equal to L. Let $\varepsilon > 0$. There exist $\delta_- > 0$ and $\delta_+ > 0$ such that $a - \delta_- < x < a$ implies $|f(x) - L| < \varepsilon$ and $a < x < a + \delta_+$ implies $|f(x) - L| < \varepsilon$. Set

$$\delta = \min\{\delta_-, \delta_+\}.$$

If $0 < |x - a| < \delta$, then either $x < a$ or $x > a$. In both cases, $|f(x) - L| < \varepsilon$. Hence $\lim_{x\to a} f(x) = L$. $\square$

At an endpoint of an interval, only the one-sided limit that lies in the domain is relevant. For a function on $[a, b]$, for example, we approach a from the right and b from the left.

Example 4.10. Consider

$$f(x) = \begin{cases} x^2, & x < 1, \\ 2x + 1, & x \geqslant 1. \end{cases}$$

Then

$$\lim_{x\to 1^-} f(x) = 1, \qquad \lim_{x\to 1^+} f(x) = 3.$$

Since the two values are different, $\lim_{x\to 1} f(x)$ does not exist. Notice that $f(1) = 3$, but this does not repair the disagreement from the two sides.

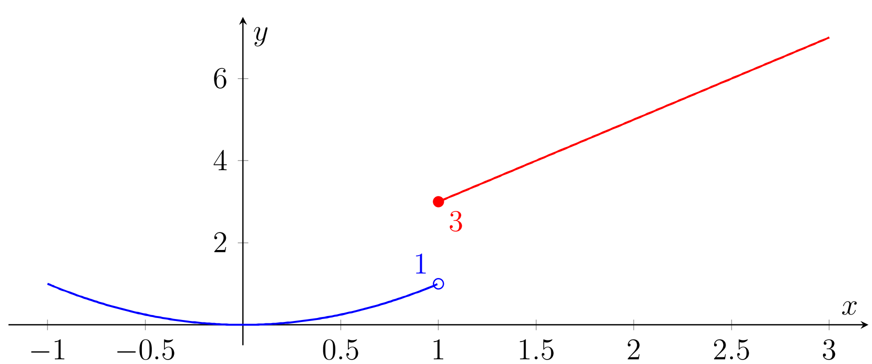

Figure 19: Different one-sided limits produce a jump at $x = 1$.

4.4 Infinite limits and limits at infinity

We also use limits to describe unbounded behaviour. Writing

$$\lim_{x \to a} f(x) = +\infty$$

means that $f(x)$ becomes larger than any prescribed number when x is sufficiently close to a. The symbol $+\infty$ is not a real number; this notation describes a kind of divergence.

Example 4.11. For $f(x) = 1/(x-2)$,

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty, \qquad \lim_{x \to 2^+} \frac{1}{x-2} = +\infty.$$

The two-sided limit does not exist, even as an infinite limit, because the signs are different.

The notation $x \to +\infty$ means that x increases without bound. For rational functions, dividing numerator and denominator by the highest relevant power of x is often the simplest method.

Keep in mind the following hierarchy as $x \to +\infty$: positive powers of x grow faster than logarithms, while exponentials grow faster than every fixed power. For $p > 0$ and $a > 1$,

$$\lim_{x \to +\infty} \frac{\ln x}{x^p} = 0, \qquad \lim_{x \to +\infty} \frac{x^p}{a^x} = 0.$$

These comparisons are often called a scale of growth. We will justify them later using derivative methods. For now, they may be used as standard limits.

Example 4.12. Let us calculate

$$\lim_{x \to +\infty} \frac{3x^2 - \sin x}{2x^2 + 1}.$$

Dividing by x^2, we obtain

$$\frac{3 - \sin(x)/x^2}{2 + 1/x^2}.$$

The quotient $\sin(x)/x^2$ tends to zero because its numerator is bounded. Hence,

$$\lim_{x \to +\infty} \frac{3x^2 - \sin x}{2x^2 + 1} = \frac{3}{2}.$$

4.5 Indeterminate forms

If direct substitution produces $0/0$, we have not found the limit. We have only found an **indeterminate form**. The expression may approach any real number, may diverge to infinity or may have no limit at all. The common indeterminate forms are

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad \infty - \infty, \qquad 0 \cdot \infty, \qquad 1^\infty, \qquad 0^0, \qquad \infty^0.$$

They are warnings, not answers. At this stage we remove them by algebraic simplification, rationalization, comparison, or standard limits. L'Hospital's rule will be introduced later, after the Mean Value Theorem.

Some forms can be converted into others. A product $f(x)g(x)$ of the form $0 \cdot \infty$ can be rewritten as

$$\frac{f(x)}{1/g(x)} \qquad \text{or} \qquad \frac{g(x)}{1/f(x)},$$

depending on which quotient is easier. A difference $\infty - \infty$ may become a single quotient after combining fractions, or it may simplify after rationalization. For a power $f(x)^{g(x)}$, taking logarithms changes the problem into the product $g(x)\ln f(x)$. Identifying the form tells us which transformation to try.

Example 4.13. Consider an $\infty - \infty$ form:

$$\lim_{x\to+\infty} \left(\sqrt{x^2+3x} - x\right).$$

Multiplying and dividing by the conjugate gives

$$\begin{aligned} \sqrt{x^2+3x} - x &= \frac{(x^2+3x)-x^2}{\sqrt{x^2+3x}+x} \\ &= \frac{3x}{\sqrt{x^2+3x}+x} = \frac{3}{\sqrt{1+3/x}+1}. \end{aligned}$$

Therefore, the limit is $3/2$.

4.6 Standard limits

The following limits are basic tools. They will also allow us to calculate the derivatives of elementary functions.

Fact 4.14 (Standard limits)**.** All trigonometric limits below use radians. We have

$$\lim_{x\to0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to0}\frac{1-\cos x}{x^2} = \frac{1}{2}, \qquad \lim_{x\to0}\frac{\tan x}{x} = 1,$$
$$\lim_{x\to0}\frac{e^x-1}{x} = 1, \qquad \lim_{x\to0}\frac{\ln(1+x)}{x} = 1, \qquad \lim_{x\to0}(1+x)^{1/x} = e.$$

More generally, for $a > 0$,

$$\lim_{x\to0}\frac{a^x-1}{x} = \ln a.$$

The first trigonometric limit follows geometrically from the squeeze theorem. The use of radians is essential: with degrees, the limit would not be 1.

Example 4.15. Let us calculate

$$\lim_{x\to0}\frac{e^{2x}-1}{\sin(3x)}.$$

We insert the quantities needed for the standard limits:

$$\frac{e^{2x}-1}{\sin(3x)} = 2\frac{e^{2x}-1}{2x}\cdot\frac{3x}{\sin(3x)}\cdot\frac{1}{3}.$$

Both middle quotients tend to 1, so the required limit is $2/3$.

Example 4.16. The standard logarithmic limit also applies away from the origin. Consider

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Put $u = x - 1$. Then $u \to 0$, $x = 1 + u$, and

$$\frac{\ln x}{x - 1} = \frac{\ln(1 + u)}{u} \longrightarrow 1.$$

This limit will soon give the derivative of $\ln x$ at $x = 1$.

Example 4.17. The expression

$$\lim_{x \to 0} (1 + 3x)^{2/x}$$

has the form 1^∞. Put $u = 3x$. Then $u \to 0$ and

$$(1 + 3x)^{2/x} = \left((1 + u)^{1/u}\right)^6.$$

Consequently, the limit is e^6.