5 Lecture #4: Week 2

5.1 Continuity at a point

Informally, a function is continuous if its graph can be drawn locally without lifting the pencil. This picture is useful, but it is not a definition. A graph may be too complicated to draw, and at an endpoint only one side is available. Limits give the precise language we need.

Definition 5.1. Let $f : A \to \mathbb{R}$ and let $a \in A$. The function f is **continuous at** a if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in A, \quad |x - a| < \delta \quad \Longrightarrow \quad |f(x) - f(a)| < \varepsilon.$$

If a can be approached by other points of A, this is equivalent to

$$\lim_{x \to a} f(x) = f(a),$$

where only approaches through points of A are considered. The function is continuous on a set if it is continuous at every point of that set.

For continuity at an interior point, three conditions must hold:

(1) $f(a)$ is defined;

(2) $\lim_{x \to a} f(x)$ exists;

(3) the limit equals $f(a)$.

At the left endpoint of an interval we use the right-hand limit, and at the right endpoint we use the left-hand limit. Thus, $\sqrt{x}$ is continuous at 0 as a function on $[0, +\infty)$ because $\lim_{x \to 0^+} \sqrt{x} = 0$.

Example 5.2. Determine whether

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2, \\ 7, & x = 2 \end{cases}$$

is continuous at 2. For $x \neq 2$, $f(x) = x + 2$, and therefore

$$\lim_{x \to 2} f(x) = 4.$$

Since $f(2) = 7$, the function is not continuous at 2. If we replace the value 7 by 4, the resulting function is continuous there.

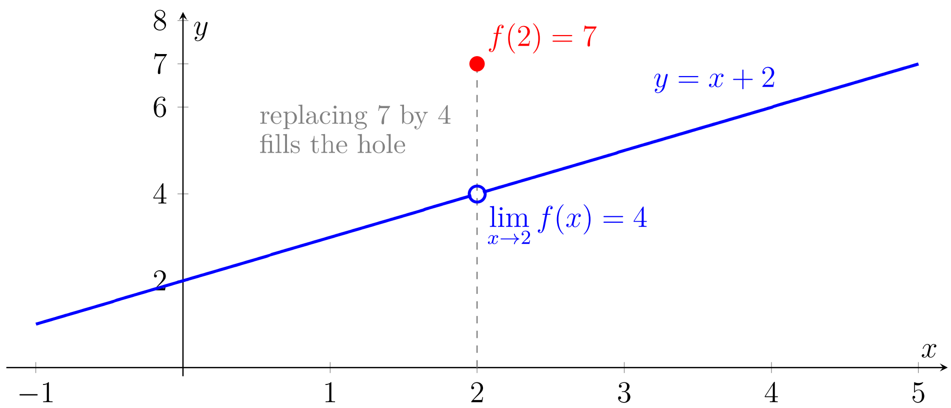

Figure 20: The graph approaches the open point $(2, 4)$, but the actual value is $f(2) = 7$. Hence f is not continuous at 2.

The identity function and every constant function are continuous. The limit laws and the chain rule for limits give the following result.

Theorem 5.3 (Algebra of continuous functions)**.** If f and g are continuous at a, then $f+g$, $f-g$ and fg are continuous at a. If $g(a) \neq 0$, then f/g is continuous at a. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Consequently, polynomials are continuous on $\mathbb{R}$, rational functions are continuous where their denominators do not vanish, and the exponential, logarithmic, trigonometric and inverse trigonometric functions are continuous on their natural domains. In practical calculations, this is why direct substitution works so often.

The local character of continuity is also useful. To decide whether a piecewise function is continuous at a point strictly inside one piece, the other formulas are irrelevant. Only junction points require a comparison of formulas. At a junction a, calculate the one-sided limits and compare them with $f(a)$.

Example 5.4. The function

$$f(x) = \ln\left(2 + \sqrt{1 + x^2}\right)$$

is continuous on $\mathbb{R}$. Indeed, $1 + x^2 > 0$, the square-root function is continuous on $[0, +\infty)$, and its value is at least 1. Hence the logarithm receives an argument at least 3, so every operation and composition is valid.

Example 5.5. Find c so that

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ c, & x = 0 \end{cases}$$

is continuous at 0. The standard limit gives

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Continuity requires this limit to equal $f(0) = c$, so the unique choice is $c = 1$. This is a typical continuous extension: we fill a removable hole with its limiting value.

5.2 Bolzano and the Intermediate Value Theorem

A continuous function cannot move from a negative value to a positive value without passing through zero. This simple geometric idea is one of the most useful existence principles in Calculus.

Theorem 5.6 (Bolzano's theorem)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous. If $f(a)f(b) < 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

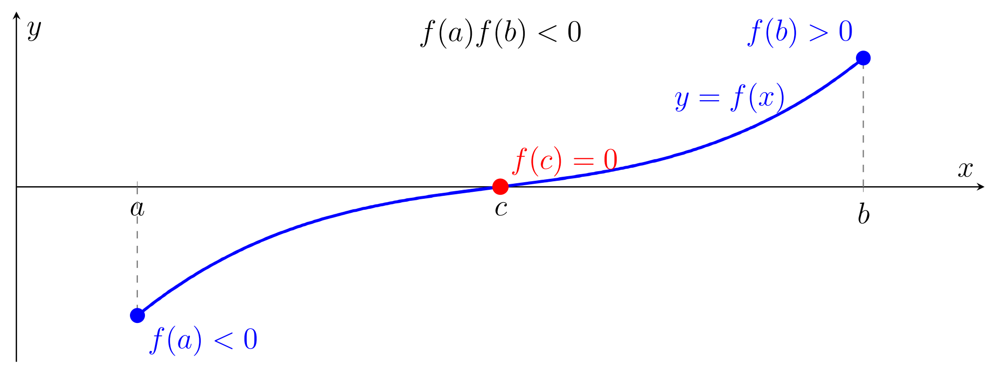

Figure 21: A continuous function whose values at the endpoints have opposite signs must cross the x-axis at some point $c \in (a,b)$.

Bolzano's theorem guarantees existence, but it does not say that the zero is unique, and it does not provide an exact formula for it. These are separate questions.

It is equally important to know when the theorem cannot be used. If $f(a)$ and $f(b)$ have the same sign, the function may still have a zero between them, but Bolzano gives no conclusion. If continuity fails, a sign change may occur by a jump without any zero. For example, the function that equals -1 for $x < 0$ and 1 for $x \geqslant 0$ changes sign but never vanishes.

Example 5.7. Consider the equation

$$x^5 + x = 1.$$

Define $f(x) = x^5 + x - 1$. This polynomial is continuous, and

$$f(0) = -1 < 0, \qquad f(1) = 1 > 0.$$

Therefore, there is at least one solution in $(0,1)$. We have proved existence without having an algebraic formula for the solution.

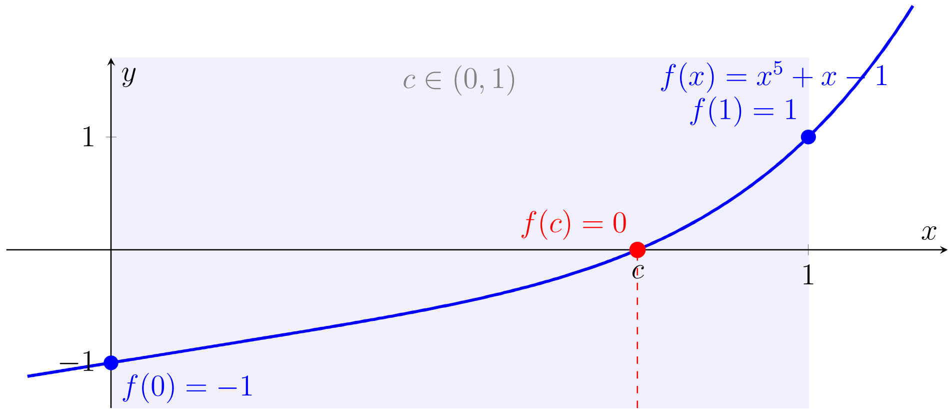

Figure 22: Since $f(0) < 0 < f(1)$ and f is continuous, its graph crosses the x-axis at some $c \in (0,1)$. Numerically, $c \approx 0.755$.

The same principle applies to every intermediate height, not only zero.

Theorem 5.8 (Intermediate Value Theorem)**.** Let I be an interval and let $f : I \to \mathbb{R}$ be continuous. If $x_1, x_2 \in I$ and a number y lies between $f(x_1)$ and $f(x_2)$, then there exists a point c between x_1 and x_2 such that $f(c) = y$. Equivalently, the image $f(I)$ is an interval.

The assumption that the domain is an interval is essential. A continuous function on two separated pieces can have an image with a gap.

Example 5.9. Suppose that the temperature along a metal rod is described by a continuous function $T : [0,1] \to \mathbb{R}$, with $T(0) = 20$ and $T(1) = 80$. The Intermediate Value Theorem guarantees that, for every temperature between 20 and 80 degrees, there is at least one point on the rod having that temperature. The theorem does not tell us where the point is or whether it is unique.

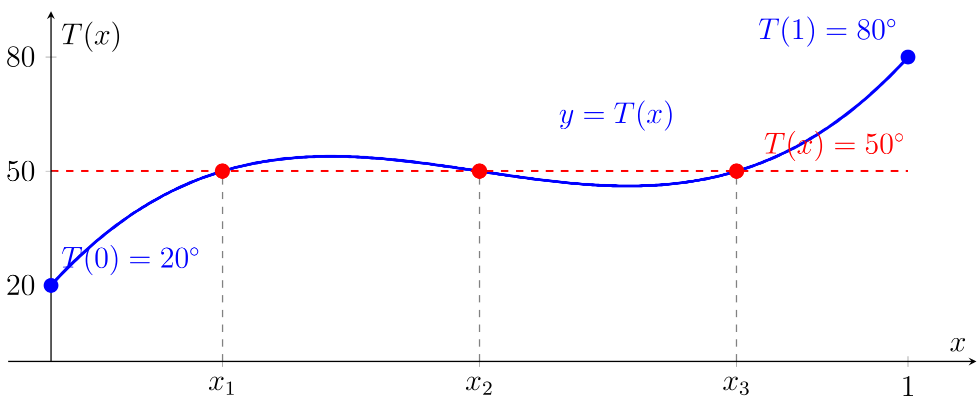

Figure 23: The Intermediate Value Theorem guarantees that every temperature between 20° and 80° occurs somewhere along the rod. An intermediate temperature need not occur at a unique point.

5.3 The Extreme Value Theorem

Continuity on a closed bounded interval gives more than intermediate values: it also prevents the graph from escaping to infinity or approaching an unattained endpoint value.

Theorem 5.10 (Extreme Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains both an absolute minimum and an absolute maximum. Thus, there exist $x_m, x_M \in [a, b]$ such that

$$f(x_m) \leqslant f(x) \leqslant f(x_M) \qquad \text{for every } x \in [a, b].$$

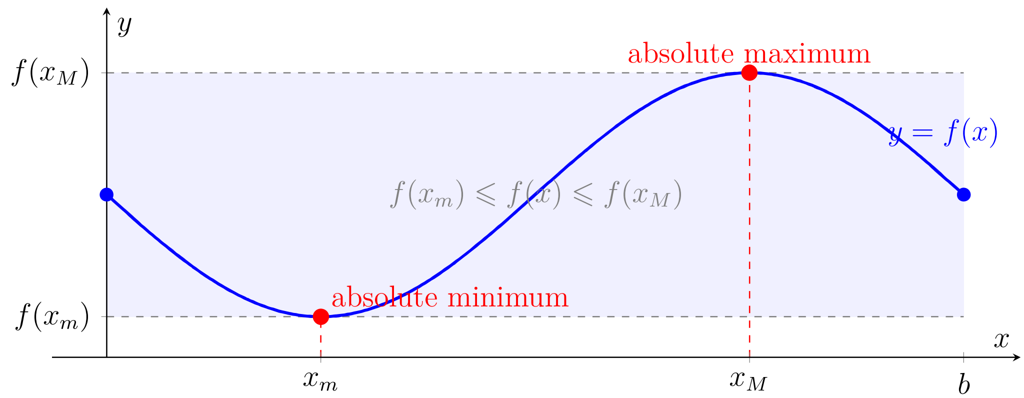

Figure 24: A continuous function on the closed interval $[a, b]$ is bounded and attains both its absolute minimum and its absolute maximum.

All hypotheses matter. The function $1/x$ is continuous on $(0, 1]$ but is not bounded there; the interval is not closed. The function $f(x) = x$ on $(0, 1)$ is bounded but has neither a maximum nor a minimum; again, the endpoints are missing.

Example 5.11. Consider $f(x) = |2x - 1|$ on $[0, 1]$. The function is continuous, so the Extreme Value Theorem guarantees that its extrema are attained. Here they can be found directly: the absolute value is smallest when $2x - 1 = 0$, namely at $x = 1/2$, and

$$f(1/2) = 0.$$

At the endpoints, $f(0) = f(1) = 1$, which is the maximum. The theorem guarantees existence before we know how to locate the points; later, derivatives will give a systematic method for finding them.

5.4 Secant slopes and the tangent problem

Limits also answer a geometric question. Let $y = f(x)$ and fix a point $P = (a, f(a))$ on the graph. A second point $Q = (a + h, f(a + h))$ determines a secant line. Its slope is

$$\frac{f(a + h) - f(a)}{h}.$$

As $h \to 0$, the point Q moves toward P. If the secant slopes approach a finite number, that number is the slope of the tangent line at P.

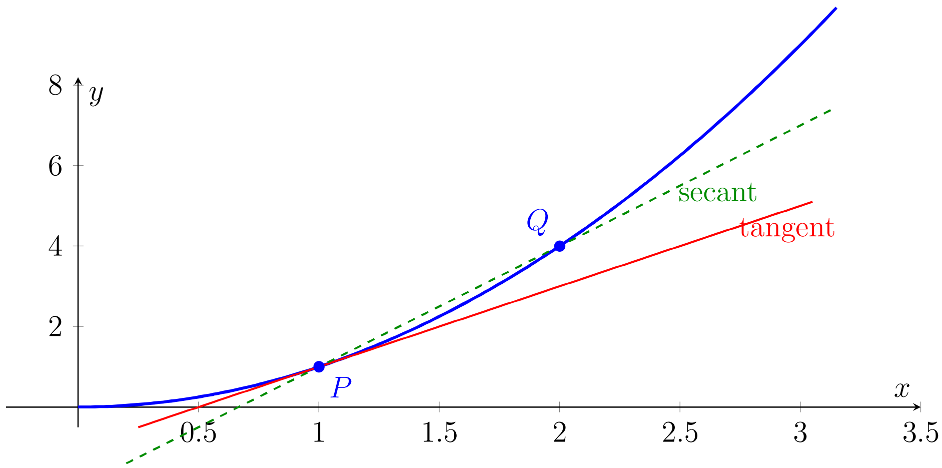

Figure 25: For $f(x) = x^2$ at $x = 1$, secant slopes approach the tangent slope 2.

For $f(x) = x^2$ and $a = 1$, the secant slope is

$$\frac{(1+h)^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h.$$

It approaches 2 as $h \to 0$. Hence the tangent line through $(1, 1)$ has equation

$$y - 1 = 2(x - 1), \qquad \text{or} \qquad y = 2x - 1.$$

5.5 Velocity and rates of change

Suppose that $s(t)$ gives the position of a moving object at time t. Over a time interval from t to $t + h$, its average velocity is

$$\frac{s(t+h) - s(t)}{h}.$$

If this quotient has a limit as $h \to 0$, we call it the instantaneous velocity at time t. The same construction measures an instantaneous rate of change whenever one quantity depends on another: temperature with respect to time, cost with respect to production, or voltage with respect to current.

Average and instantaneous rates answer different questions. A car may have average speed 80 kilometres per hour over a journey while being stopped at one instant and travelling faster than 80 at another. The limit shrinks the observation interval around one chosen time, which is why it captures the instantaneous behaviour.

Example 5.12. Let the height of a ball be $s(t) = 20t - 5t^2$ metres. Its average velocity from $t = 1$ to $t = 1 + h$ is

$$\begin{aligned} \frac{s(1+h) - s(1)}{h} &= \frac{20(1+h) - 5(1+h)^2 - 15}{h} \\ &= \frac{10h - 5h^2}{h} = 10 - 5h. \end{aligned}$$

As $h \to 0$, this approaches 10 metres per second. In the next lecture we will give this limiting rate a name: the derivative.