Calculus 1

Lecture 3: Limits

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- Explain what a limit means using ε and δ.
- Calculate limits using algebra, comparison and standard limits.
- Calculate one-sided and infinite limits.
- Recognize and remove basic indeterminate forms.
- Important theorems.

What a limit measures

A limit describes behaviour near a point

Consider

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A limit concerns nearby values, not necessarily the value at the point.

Factorization reveals the nearby behaviour

Whenever $x \neq 1$,

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Whenever $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1}$$

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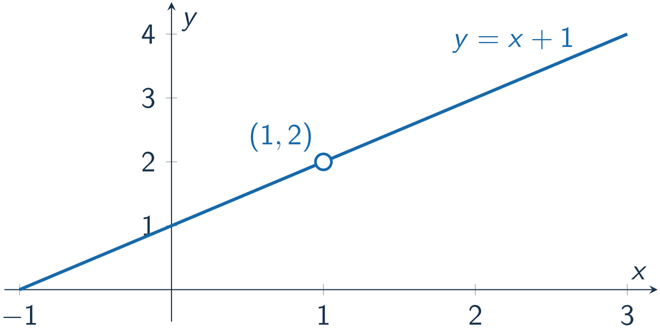
Whenever $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1.$$

Therefore, as x approaches 1, the value of the function approaches

$$1 + 1 = 2.$$

The missing value does not prevent the limit



$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Function value and limit answer different questions

Function value

$f(a)$ describes what happens **at** the point.

Limit

$\lim_{x \to a} f(x)$ describes what happens **around** the point.

A limit may exist even when $f(a)$ is undefined or has a different value.

Sequences provide another way to test a limit

Suppose $x_n \neq a$ and $x_n \to a$.

If

$$\lim_{x \to a} f(x) = L,$$

then every such sequence must satisfy

$$f(x_n) \longrightarrow L.$$

Consequently, two approaches producing different limiting values prove that the limit does not exist.

Two sequences expose the oscillation of $\sin(1/x)$

Let

$$x_n = \frac{1}{\pi/2 + 2\pi n}, \qquad y_n = \frac{1}{3\pi/2 + 2\pi n}.$$

Both sequences tend to 0, but

$$\sin(1/x_n) = 1, \qquad \sin(1/y_n) = -1.$$

$\lim_{x \to 0} \sin(1/x)$ does not exist.

Changing one value cannot change a limit

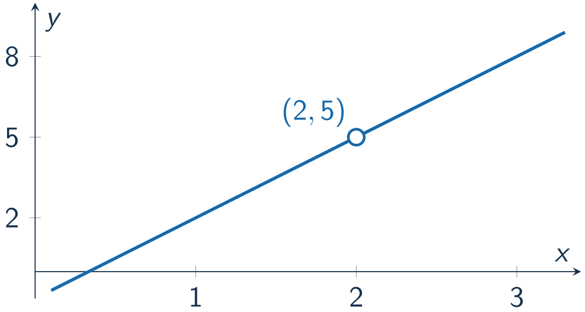
Let

$$f(x) = \begin{cases} 3x - 1, & x \neq 2, \\ 100, & x = 2. \end{cases}$$

Near $x = 2$, but away from 2, the function agrees exactly with $3x - 1$.

$$3x - 1 \longrightarrow 5 \qquad \text{as} \qquad x \longrightarrow 2.$$

The isolated value is irrelevant



At the point

$$f(2) = 100.$$

Around the point

$$f(x) = 3x - 1.$$

One exceptional value cannot affect nearby behaviour

$$\lim_{x \to 2} f(x) = 5, \qquad f(2) = 100.$$

The limit is 5, even though the function value is 100.

The epsilon-delta definition

The definition turns closeness into inequalities

We write

$$\lim_{x \to a} f(x) = L$$

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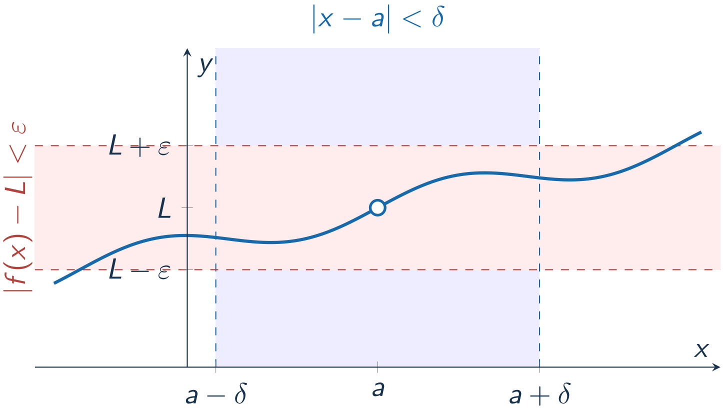
$$\lim_{x \to a} f(x) = L$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in A, \qquad 0 < |x - a| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon.$$

ε controls the output; δ controls the input.

A delta-window must enter the epsilon-band



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- *For every* $\varepsilon > 0$: the requested accuracy may be arbitrarily small.
- *There exists* $\delta > 0$: the input window may depend on ε.
- $0 < |x - a|$: the value at a is excluded.
- $|f(x) - L| < \varepsilon$: all nearby outputs lie in the epsilon-band.

Example: prove a linear limit directly

We want to prove

$$\lim_{x \to 3} (2x - 1) = 5.$$

Given $\varepsilon > 0$, our target is

$$|(2x - 1) - 5| < \varepsilon.$$

The question is: how small should $|x - 3|$ be?

Rewrite the output error using the input error

$$\begin{aligned} |(2x-1)-5| &= |2x-6| \\ &= 2\,|x-3|\,. \end{aligned}$$

Therefore,

$$2\,|x-3| < \varepsilon$$

is guaranteed whenever

$$|x-3| < \frac{\varepsilon}{2}.$$

The calculation tells us how to choose delta

Choose

$$\delta = \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} 0 < |x-3| < \delta \quad \implies \quad & |(2x-1)-5| \\ & = 2\,|x-3| \\ & < 2\delta = \varepsilon. \end{aligned}$$

$$\lim_{x \to 3} (2x-1) = 5.$$

An epsilon-delta proof follows a reusable pattern

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3. Choose δ from that estimate.
4. Verify the implication from the definition.

Work backwards to find δ, then write the proof forwards.

Limit laws and squeezing

Finite limits obey the usual arithmetic rules

If

$$f(x) \longrightarrow L, \qquad g(x) \longrightarrow M,$$

then

$$\begin{aligned} f(x) + g(x) &\longrightarrow L + M, \\ f(x)g(x) &\longrightarrow LM, \\ \frac{f(x)}{g(x)} &\longrightarrow \frac{L}{M} \qquad (M \neq 0). \end{aligned}$$

Constants, integer powers and admissible roots also pass through limits.

Polynomials and rational functions allow substitution

For every polynomial P,

$$\lim_{x \to a} P(x) = P(a).$$

For a rational function,

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$$

provided that

$$Q(a) \neq 0.$$

Direct substitution works when the denominator survives

$$\lim_{x \to 2} \frac{x^3 + 4x - 1}{x^2 + 1} = \frac{2^3 + 4 \cdot 2 - 1}{2^2 + 1}$$
$$= \frac{15}{5}$$
$$= 3.$$

There is no indeterminate form because the denominator tends to 5, not to 0.

The squeeze theorem traps a difficult function

Suppose that, near a,

$$g(x) \leq f(x) \leq h(x),$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then

$$\lim_{x\to a} f(x) = L.$$

Bounded oscillation can still have a limit

Since

$$|\sin(1/x)| \leq 1,$$

we have

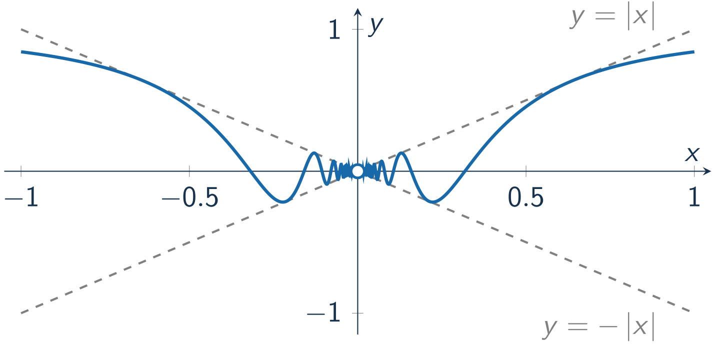
$$|x \sin(1/x)| \leq |x| \, .$$

Equivalently,

$$-|x| \leq x \sin(1/x) \leq |x| \, .$$

Both outer functions tend to 0 as $x \to 0$.

The oscillations remain inside a shrinking envelope



$$\lim_{x \to 0} x \sin(1/x) = 0.$$

One-sided limits

One-sided limits remember the direction of approach

The left-hand limit

$$\lim_{x \to a^-} f(x)$$

uses only points $x < a$.

The right-hand limit

$$\lim_{x \to a^+} f(x)$$

uses only points $x > a$.

A two-sided limit requires agreement from both sides

If the domain approaches a from both sides, then

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

The left and right limits must both exist and be equal.

At an endpoint only one direction is available

For a function defined on $[a, b]$:

- At a, we approach from the right:

$$\lim_{x \to a^+} f(x).$$

- At b, we approach from the left:

$$\lim_{x \to b^-} f(x).$$

Example: inspect each branch separately

Consider

$$f(x) = \begin{cases} x^2, & x < 1, \\ 2x + 1, & x \geq 1. \end{cases}$$

To find the limit at 1, use:

- x^2 on the left of 1;
- $2x + 1$ on the right of 1.

The two branches approach different values

From the left,

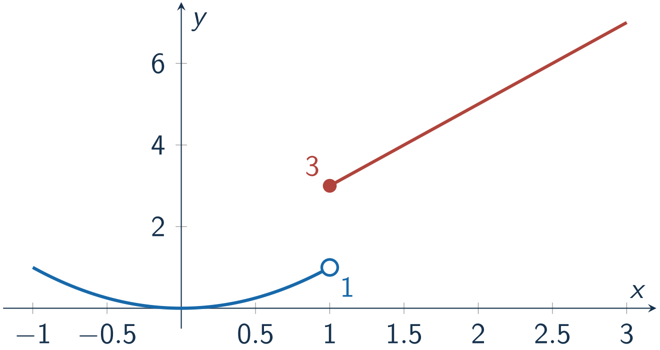
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1.$$

From the right,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x+1) = 3.$$

The two values do not agree.

The graph displays a jump at $x = 1$



$$\lim_{x \to 1} f(x) \quad \text{does not exist.}$$

The value at the jump cannot repair the limit

For the piecewise function,

$$f(1) = 3.$$

However,

$$\lim_{x \to 1^-} f(x) = 1 \neq 3 = \lim_{x \to 1^+} f(x).$$

A two-sided limit fails whenever the one-sided limits disagree.

Infinite limits and limits at infinity

An infinite limit describes unbounded behaviour

The notation

$$\lim_{x \to a} f(x) = +\infty$$

means that $f(x)$ becomes larger than every prescribed positive number when x is sufficiently close to a.

$+\infty$ is not a real number; it describes a type of divergence.

A reciprocal changes sign across its vertical asymptote

For

$$f(x) = \frac{1}{x-2},$$

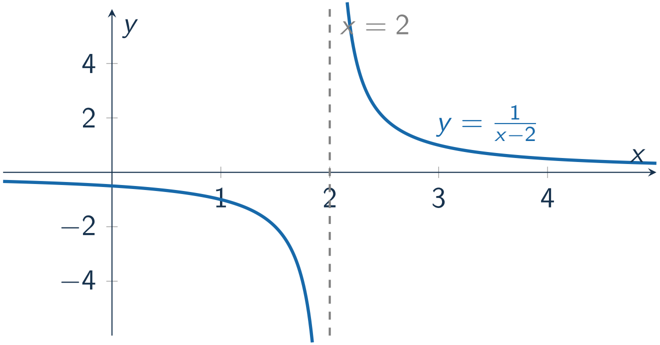
we have

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

and

$$\lim_{x\to 2^+} \frac{1}{x-2} = +\infty.$$

The graph separates the two infinite behaviours



Opposite signs prevent a two-sided infinite limit

$$\lim_{x\to 2^-} \frac{1}{x-2} = -\infty, \qquad \lim_{x\to 2^+} \frac{1}{x-2} = +\infty.$$

The two-sided limit does not exist.

Even for infinite limits, the two sides must describe the same behaviour.

A limit at infinity describes long-term behaviour

The notation

$$x \to +\infty$$

means that x increases without bound.

For a rational expression, divide the numerator and denominator by the highest relevant power of x.

This reveals which terms survive and which terms tend to zero.

The main families have a hierarchy of growth

For every $p > 0$ and $a > 1$,

$$\lim_{x \to +\infty} \frac{\ln x}{x^p} = 0,$$

so powers grow faster than logarithms, and

$$\lim_{x \to +\infty} \frac{x^p}{a^x} = 0,$$

so exponentials grow faster than powers.

$$\text{logarithm} \quad < \quad \text{power} \quad < \quad \text{exponential.}$$

Example: divide by the dominant power

Consider

$$\lim_{x \to +\infty} \frac{3x^2 - \sin x}{2x^2 + 1}.$$

Dividing numerator and denominator by x^2 gives

$$\frac{3 - \sin(x)/x^2}{2 + 1/x^2}.$$

It remains to understand the two small terms.

Boundedness removes the oscillating term

Since $|\sin x| \leq 1$,

$$0 \leq \frac{|\sin x|}{x^2} \leq \frac{1}{x^2}.$$

As $x \to +\infty$,

$$\frac{1}{x^2} \longrightarrow 0.$$

Therefore, by squeezing,

$$\frac{\sin x}{x^2} \longrightarrow 0.$$

Only the leading coefficients remain

Now

$$\frac{3 - \sin(x)/x^2}{2 + 1/x^2} \longrightarrow \frac{3 - 0}{2 + 0}.$$

$$\lim_{x \to +\infty} \frac{3x^2 - \sin x}{2x^2 + 1} = \frac{3}{2}.$$

Indeterminate forms

An indeterminate form is a signal, not an answer

Substitution may produce one of the symbolic forms

$$\frac{0}{0}, \qquad \frac{\infty}{\infty}, \qquad \infty - \infty, \qquad 0 \cdot \infty,$$

or, for powers,

$$1^{\infty}, \qquad 0^{0}, \qquad \infty^{0}.$$

These forms do not determine the limit. They tell us that the expression must be transformed before a limit law can be applied.

One $0/0$ form, several possible limits

As $x \to 0$,

$$\frac{x}{x} \longrightarrow 1, \qquad \frac{x^2}{x} \longrightarrow 0, \qquad \frac{|x|}{x} \text{ has no two-sided limit.}$$

All three expressions initially have the same form $0/0$, but their limits are different.

An indeterminate form describes missing information, not a value.

Choose a transformation that matches the expression

Algebraic expressions

- factor and cancel;
- take a common denominator;
- divide by the dominant power;
- multiply by a conjugate.

Products and powers

- rewrite a product as a quotient;
- use logarithms for variable powers;
- use a standard limit;
- squeeze when a factor is bounded.

Example: subtraction creates an $\infty - \infty$ form

Evaluate

$$\lim_{x \to +\infty} \left(\sqrt{x^2 + 3x} - x \right).$$

Direct inspection gives the form

$$\infty - \infty.$$

The square root suggests multiplying by the conjugate.

Multiply by the conjugate

$$\begin{aligned}\sqrt{x^2+3x}-x &= \left(\sqrt{x^2+3x}-x\right)\frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x}\\ &= \frac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}\\ &= \frac{3x}{\sqrt{x^2+3x}+x}.\end{aligned}$$

The subtraction has disappeared.

Divide numerator and denominator by x

For $x > 0$,

$$\frac{3x}{\sqrt{x^2+3x}+x} = \frac{3}{\sqrt{1+3/x}+1}.$$

Now ordinary substitution is valid:

$$\frac{3}{\sqrt{1+3/x}+1} \longrightarrow \frac{3}{1+1}.$$

$$\lim_{x\to+\infty}\left(\sqrt{x^2+3x}-x\right) = \frac{3}{2}.$$

Standard limits

A small collection of limits solves many problems

As $x \to 0$,

$$\frac{\sin x}{x} \longrightarrow 1, \qquad \frac{1 - \cos x}{x^2} \longrightarrow \frac{1}{2}, \qquad \frac{\tan x}{x} \longrightarrow 1.$$

These trigonometric formulas require angles to be measured in **radians**.

Exponential and logarithmic limits

As $x \to 0$,

$$\frac{e^x - 1}{x} \longrightarrow 1, \qquad \frac{a^x - 1}{x} \longrightarrow \ln a \quad (a > 0),$$

and

$$\frac{\ln(1+x)}{x} \longrightarrow 1.$$

For variable powers,

$$(1+x)^{1/x} \longrightarrow e.$$

Example: identify two copies of $\sin u/u$

Evaluate

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}.$$

Both numerator and denominator tend to zero. Insert the useful factors $2x$ and $3x$:

$$\frac{e^{2x}-1}{\sin(3x)} = \frac{e^{2x}-1}{2x}\,\frac{3x}{\sin(3x)}\,\frac{2}{3}.$$

Apply the standard limits factor by factor

When $x \to 0$, both $2x \to 0$ and $3x \to 0$. Hence

$$\frac{e^{2x} - 1}{2x} \longrightarrow 1, \qquad \frac{3x}{\sin(3x)} \longrightarrow 1.$$

Therefore,

$$\frac{e^{2x} - 1}{\sin(3x)} \longrightarrow 1 \cdot 1 \cdot \frac{2}{3}.$$

$$\lim_{x \to 0} \frac{e^{2x} - 1}{\sin(3x)} = \frac{2}{3}.$$

Example: shift the logarithmic standard limit

Evaluate

$$\lim_{x \to 1} \frac{\ln x}{x - 1}.$$

Set

$$u = x - 1.$$

Then $u \to 0$ and $x = 1 + u$, so

$$\frac{\ln x}{x - 1} = \frac{\ln(1 + u)}{u}.$$

The substitution reveals the known pattern

The standard limit gives

$$\frac{\ln(1+u)}{u} \longrightarrow 1 \qquad (u \to 0).$$

Therefore,

$$\lim_{x\to 1} \frac{\ln x}{x-1} = 1.$$

The variable name does not matter; the local pattern does.

Example: a variable power

Evaluate

$$\lim_{x \to 0} (1 + 3x)^{2/x}.$$

The base tends to 1 while the exponent becomes unbounded, so this has the indeterminate form 1^∞.

The model limit is

$$(1 + u)^{1/u} \longrightarrow e.$$

Rewrite the expression around the model limit

Set $u = 3x$. Then $u \to 0$ and

$$\frac{2}{x} = \frac{6}{u}.$$

Thus

$$\begin{aligned}(1+3x)^{2/x} &= (1+u)^{6/u} \\ &= \left((1+u)^{1/u}\right)^6.\end{aligned}$$

Taking the limit gives

$$\left((1+u)^{1/u}\right)^6 \longrightarrow e^6.$$

$$\lim_{x\to 0}(1+3x)^{2/x} = e^6.$$

Closing summary

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4. Compare with a standard limit or use the squeeze theorem.
5. Check the two sides separately near jumps and asymptotes.

Final check

Suppose

$$\lim_{x\to a^-} f(x) = 2, \qquad \lim_{x\to a^+} f(x) = 2, \qquad f(a) = 7.$$

What is $\lim_{x\to a} f(x)$?

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What is $\lim_{x \to a} f(x)$?

The one-sided limits agree, so

$$\boxed{\lim_{x \to a} f(x) = 2.}$$

The isolated value $f(a) = 7$ does not change the limit.