Calculus 1

Lecture 4: Continuity

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- ⋆ Decide whether a function is continuous at a point.
- ⋆ Use continuity to prove that values, zeros and extrema exist.
- ⋆ Recognize why each hypothesis of the main theorems matters.
- ⋆ Use a limit to obtain a tangent slope or instantaneous velocity.

Continuity at a point

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This picture does not answer every question:

- a graph may be too complicated to draw;
- a function may be defined only on one side of a;
- a precise proof needs inequalities, not a sketch.

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When a can be approached by other points of A, this is equivalent to

$$\lim_{x \to a} f(x) = f(a).$$

Three questions decide continuity at an interior point

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1. $f(a)$ is defined;
2. $\lim_{x \to a} f(x)$ exists;
3. $\lim_{x \to a} f(x) = f(a)$.

A failure of any one condition means that f is not continuous at a.

At an endpoint, only the available side matters

For a function defined on $[a, b]$,

$$f \text{ is continuous at } a \iff \lim_{x \to a^+} f(x) = f(a),$$

and

$$f \text{ is continuous at } b \iff \lim_{x \to b^-} f(x) = f(b).$$

For example, $\sqrt{x}$ is continuous at 0 on $[0, +\infty)$ because

$$\lim_{x \to 0^+} \sqrt{x} = 0 = \sqrt{0}.$$

Example: a function has a removable discontinuity

Determine whether

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2, \\ 7, & x = 2 \end{cases}$$

is continuous at 2.

We must compare the nearby behaviour with the actual value $f(2)$.

First simplify the formula away from the point

For $x \neq 2$,

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2}$$
$$= x + 2.$$

Therefore,

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 2) = 4.$$

Then compare the limit with the function value

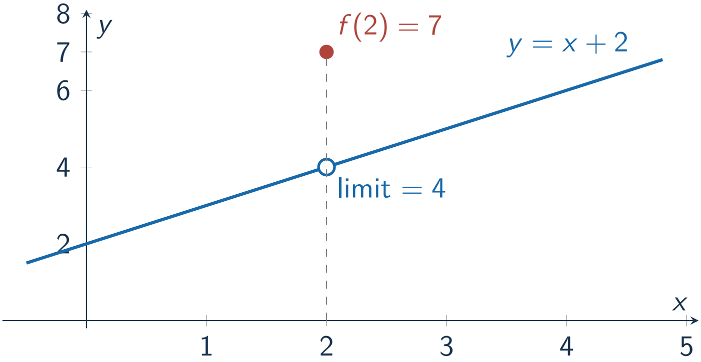
We have found

$$\lim_{x \to 2} f(x) = 4, \qquad f(2) = 7.$$

The limit exists, but it does not equal the function value.

f is not continuous at 2.

The graph approaches the hole



Filling the hole repairs the discontinuity

The nearby values already approach 4. Only the value at 2 is wrong.

Define instead

$$\widetilde{f}(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2, \\ 4, & x = 2. \end{cases}$$

Then

$$\lim_{x \to 2} \widetilde{f}(x) = 4 = \widetilde{f}(2).$$

$\widetilde{f}$ is continuous at 2.

Continuity is preserved by arithmetic operations

⋆ If f and g are continuous at a, then so are

$$f+g, \qquad f-g, \qquad fg.$$

⋆ If $g(a) \neq 0$, then f/g is also continuous at a.

⋆ If f is continuous at a and g is continuous at $f(a)$, then

$$g \circ f$$

is continuous at a.

The basic families are continuous on their domains

Consequently,

- polynomials are continuous on $\mathbb{R}$;
- rational functions are continuous where the denominator is nonzero;
- exponential and trigonometric functions are continuous on $\mathbb{R}$;
- logarithmic, inverse trigonometric and root functions are continuous on their natural domains.

This is why direct substitution works so often.

Example: verify every step of a composition

Consider

$$f(x) = \ln\left(2 + \sqrt{1 + x^2}\right).$$

For every $x \in \mathbb{R}$,

$$1 + x^2 \geq 1, \qquad \sqrt{1 + x^2} \geq 1,$$

so

$$2 + \sqrt{1 + x^2} \geq 3 > 0.$$

The logarithm therefore always receives an admissible argument.

The composition is continuous on the whole real line

The function is built from continuous operations:

$$x \longmapsto x^2 \longmapsto 1 + x^2 \longmapsto \sqrt{1 + x^2} \longmapsto 2 + \sqrt{1 + x^2} \longmapsto \ln\left(2 + \sqrt{1 + x^2}\right).$$

Every step is valid for every real x.

f is continuous on $\mathbb{R}$.

Example: choose a value that fills a trigonometric hole

Find c so that

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ c, & x = 0 \end{cases}$$

is continuous at 0.

Continuity requires

$$\lim_{x \to 0} f(x) = f(0) = c.$$

The standard limit determines the unique value

We know that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Therefore continuity at 0 requires

$$c = 1.$$

The unique continuous extension is obtained by choosing $c = 1$.

Bolzano and intermediate values

A continuous sign change must pass through zero

Suppose that f is continuous on $[a, b]$ and

$$f(a) < 0 < f(b).$$

The graph cannot jump from below the x-axis to above it.

It must pass through some point $(c, 0)$ with

$$a < c < b.$$

Bolzano's theorem guarantees a zero

Let $f : [a, b] \to \mathbb{R}$ be continuous. If

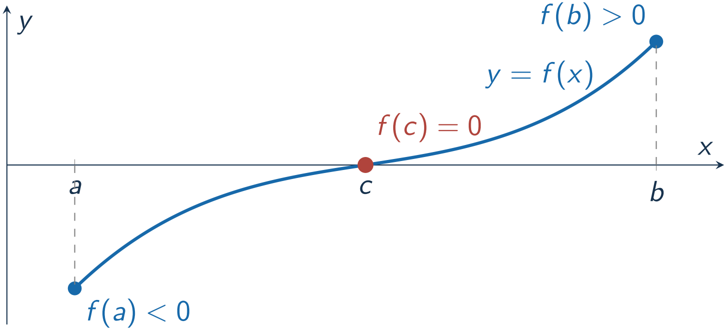
$$f(a)f(b) < 0,$$

then there exists $c \in (a, b)$ such that

$$f(c) = 0.$$

The product condition says exactly that the endpoint values have opposite signs.

Opposite endpoint signs force a crossing



The theorem proves existence, but nothing more

Bolzano's theorem guarantees that at least one zero exists in (a, b).

It does not tell us

- ⋆ where the zero is.
- ⋆ whether the zero is unique.
- ⋆ whether an exact formula for it exists.

These are separate questions.

Continuity is essential

Consider

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

The function changes sign across 0, but it never vanishes.

A jump can cross from negative to positive without taking the value 0.

The sign condition is sufficient, not necessary

If $f(a)$ and $f(b)$ have the same sign, Bolzano's theorem gives no conclusion.

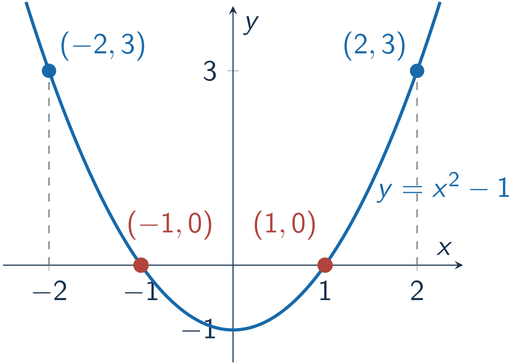
For example,

$$f(x) = x^2 - 1.$$

At the endpoints,

$$f(-2) = f(2) = 3 > 0.$$

Nevertheless, there are two zeros:



The endpoint values have the same sign, but the graph crosses the x-axis twice.

Example: prove that an equation has a solution

Consider

$$x^5 + x = 1.$$

Move every term to one side and define

$$f(x) = x^5 + x - 1.$$

The function f is a polynomial, so it is continuous on $\mathbb{R}$.

Choose an interval and test its endpoints

On the interval $[0, 1]$,

$$f(0) = 0^5 + 0 - 1 = -1 < 0,$$

$$f(1) = 1^5 + 1 - 1 = 1 > 0.$$

Hence

$$f(0)f(1) < 0.$$

All the hypotheses of Bolzano's theorem are satisfied.

Bolzano gives the existence conclusion

There exists $c \in (0, 1)$ such that

$$f(c) = 0.$$

Thus

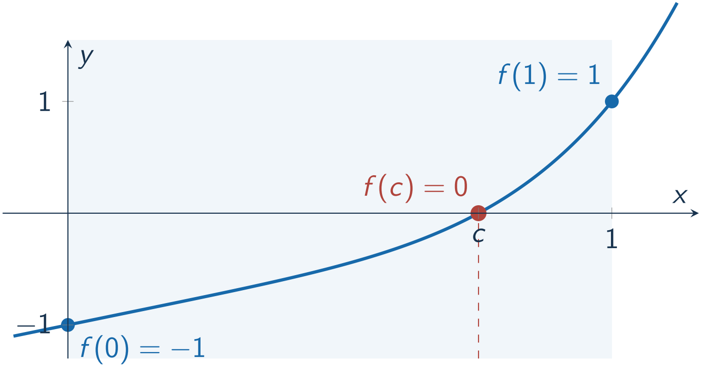
$$c^5 + c - 1 = 0,$$

or equivalently

$$c^5 + c = 1.$$

The original equation has at least one solution in $(0, 1)$.

The graph confirms the guaranteed crossing



The zero is actually unique in this example

Indeed, $f(x) = x^5 + x - 1$ is strictly increasing.

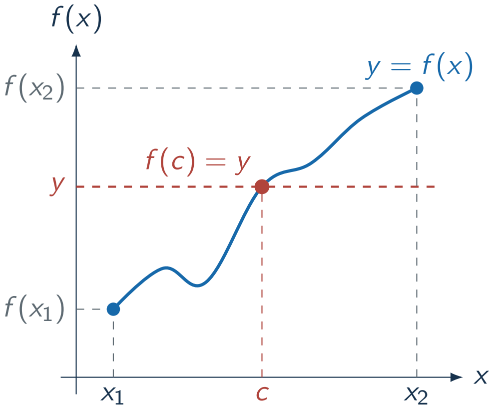
Bolzano gives existence; strict increase gives uniqueness.

Intermediate values reach every height

Let I be an interval and let $f : I \to \mathbb{R}$ be continuous. If y lies between $f(x_1)$ and $f(x_2)$, then there exists a point c between x_1 and x_2 such that $f(c) = y$.

A continuous function cannot skip an intermediate height.

Bolzano's theorem is the special case $y = 0$.



Continuous images of intervals have no gaps

The Intermediate Value Theorem is equivalent to the statement

$$f(I) \text{ is an interval whenever } I \text{ is an interval.}$$

The interval hypothesis is essential. A function can be continuous on two separated pieces while its image has a gap.

For example, the identity function on

$$[-2, -1] \cup [1, 2]$$

has the same disconnected set as its image.

Example: temperature along a metal rod

Suppose that the temperature is described by a continuous function

$$T : [0, 1] \to \mathbb{R},$$

with

$$T(0) = 20, \qquad T(1) = 80.$$

Let y be any temperature satisfying

$$20 \leq y \leq 80.$$

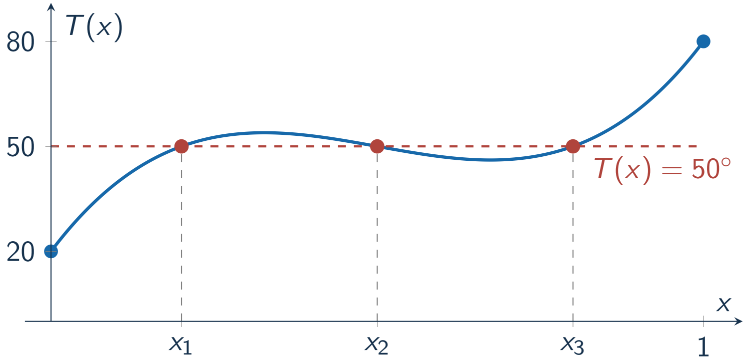
Every intermediate temperature occurs somewhere

The function T is continuous on the interval $[0, 1]$, and y lies between $T(0)$ and $T(1)$. By the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that

$$T(c) = y.$$

Every temperature from 20° to 80° occurs along the rod.

An intermediate value need not occur only once



The theorem does not locate or count the points

For the temperature 50°, the graph shows three possible positions:

$$T(x_1) = T(x_2) = T(x_3) = 50.$$

The Intermediate Value Theorem guarantees at least one point, but it does not say

- where that point is;
- how many such points there are.

Additional information about the function is needed for uniqueness.

The Extreme Value Theorem

Continuity on a closed interval controls the whole graph

A continuous function on $[a, b]$ cannot

- $\star$ escape to $+\infty$ or $-\infty$.

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- $\star$ approach a smallest value without reaching it.

The Extreme Value Theorem guarantees attained bounds

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains both an absolute minimum and an absolute maximum.

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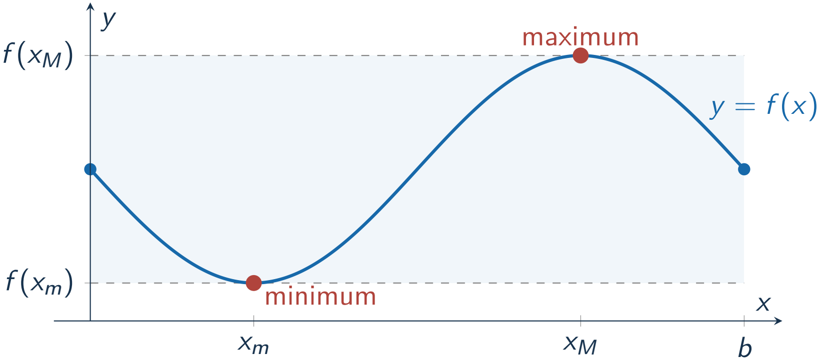
If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains both an absolute minimum and an absolute maximum. Thus there exist $x_m, x_M \in [a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M) \qquad \text{for every } x \in [a, b].$$

The points x_m and x_M may lie inside the interval or at its endpoints.

The minimum and maximum are actual function values



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If either condition is removed, boundedness or attainment can fail.

A missing endpoint can destroy boundedness

The function

$$f(x) = \frac{1}{x}$$

is continuous on $(0, 1]$.

A missing endpoint can destroy boundedness

The function

$$f(x) = \frac{1}{x}$$

is continuous on $(0, 1]$. However,

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty,$$

so f is not bounded on $(0, 1]$.

The interval is bounded, but it is not closed.

Missing endpoints may prevent attainment

The function

$$f(x) = x$$

is continuous and bounded on $(0, 1)$.

Missing endpoints may prevent attainment

The function

$$f(x) = x$$

is continuous and bounded on $(0, 1)$. But it has no minimum and no maximum there:

$$0 < f(x) < 1 \qquad \text{for every } x \in (0, 1).$$

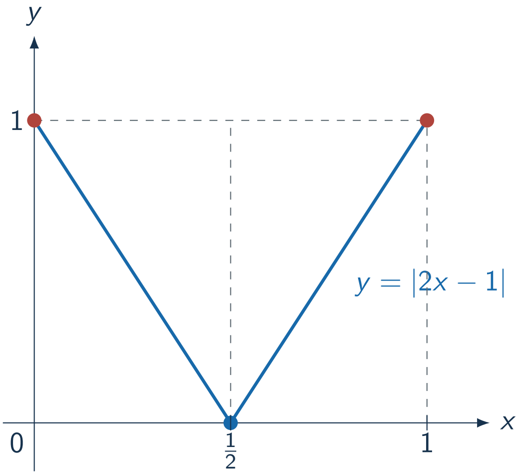
The values approach 0 and 1, but the missing endpoints prevent the function from attaining them.

Example: locate the extrema of an absolute-value function

Consider

$$f(x) = |2x - 1|$$

for $x \in [0, 1]$. The function is continuous on a closed bounded interval. Therefore, the Extreme Value Theorem guarantees that both extrema are attained.



The minimum occurs where the expression vanishes

Because an absolute value is always nonnegative,

$$|2x - 1| \geq 0.$$

The smallest possible value occurs when

$$2x - 1 = 0.$$

Thus

$$x = \frac{1}{2}, \qquad f\left(\frac{1}{2}\right) = 0.$$

The absolute minimum is 0, attained at $x = 1/2$.

The maximum is attained at both endpoints

At the endpoints,

$$f(0) = |-1| = 1, \qquad f(1) = |1| = 1.$$

For $0 \leq x \leq 1$, we have $-1 \leq 2x - 1 \leq 1$, so

$$|2x - 1| \leq 1.$$

The absolute maximum is 1, attained at $x = 0$ and $x = 1$.

The theorem guarantees existence, not locations

For $f(x) = |2x - 1|$ on $[0, 1]$,

$$\min f = 0, \qquad \max f = 1.$$

The Extreme Value Theorem guarantees that extrema exist. It does not tell us where they occur.

Derivatives will later provide a systematic method for finding them.

Secant slopes and the tangent problem

Two points determine a secant line

Fix a point on the graph of $y = f(x)$:

$$P = (a, f(a)).$$

Choose a second point

$$Q = (a + h, f(a + h)), \qquad h \neq 0.$$

The line through P and Q is a secant line.

The secant slope is a difference quotient

The horizontal change from P to Q is

$$(a+h) - a = h.$$

The vertical change is

$$f(a+h) - f(a).$$

Therefore, the secant slope is

$$\frac{f(a+h) - f(a)}{h}.$$

Moving the second point creates the tangent problem

As $h \to 0$, the point

$$Q = (a + h, f(a + h))$$

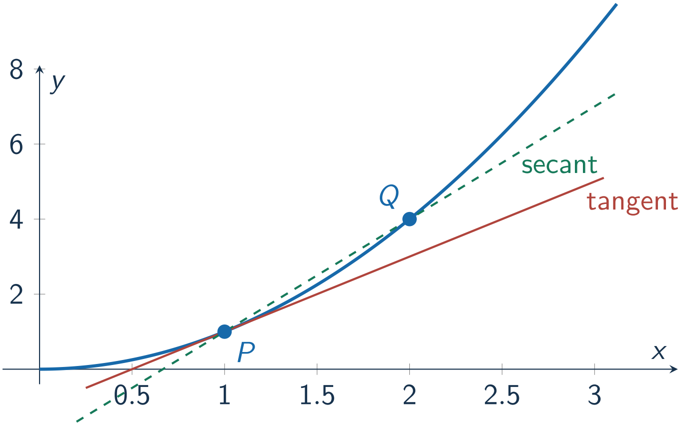
moves toward

$$P = (a, f(a)).$$

If the secant slopes approach a finite number, that number is the slope of the tangent line at P.

The tangent slope is therefore a limit.

Secant slopes approach the tangent slope



Example: calculate the secant slope for $f(x) = x^2$

Take $a = 1$. Then

$$P = (1, 1) \qquad \text{and} \qquad Q = (1 + h, (1 + h)^2).$$

The secant slope is

$$\frac{f(1 + h) - f(1)}{h} = \frac{(1 + h)^2 - 1}{h}.$$

Simplify before taking the limit

For $h \neq 0$,

$$\begin{aligned}\frac{(1+h)^2-1}{h} &= \frac{1+2h+h^2-1}{h} \\ &= \frac{2h+h^2}{h} \\ &= 2+h.\end{aligned}$$

Therefore,

$$\lim_{h\to 0}(2+h) = 2.$$

The limit gives the tangent line

The tangent slope at $(1,1)$ is

$$m = 2.$$

Using the point-slope form,

$$y - 1 = 2(x - 1).$$

Hence

$$y = 2x - 1.$$

The tangent line to $y = x^2$ at $(1,1)$ is $y = 2x - 1$.

Velocity and rates of change

Position over an interval gives average velocity

Suppose that $s(t)$ is the position of a moving object at time t.
From time t to time $t+h$, the displacement is

$$s(t+h) - s(t),$$

and the elapsed time is h.
Thus the average velocity is

$$\frac{s(t+h) - s(t)}{h}.$$

Shrinking the interval gives instantaneous velocity

If the limit exists, the instantaneous velocity at time t is

$$\lim_{h\to 0} \frac{s(t+h)-s(t)}{h}.$$

The formula has exactly the same structure as the tangent-slope limit:

$$\frac{f(a+h)-f(a)}{h}.$$

Geometry and motion lead to the same mathematical object.

The same limit measures many instantaneous rates

Whenever one quantity depends on another, a difference quotient measures an average rate of change.
Its limit can describe

- ⋆ temperature with respect to time.
- ⋆ cost with respect to production.
- ⋆ voltage with respect to current.
- ⋆ population with respect to time.

Example: the height of a ball

Let

$$s(t) = 20t - 5t^2$$

be the height of a ball in meters. Find the instantaneous velocity at $t = 1$.

Begin with the average velocity from 1 to $1 + h$:

$$\frac{s(1+h) - s(1)}{h}.$$

Calculate the two position values

At $t = 1$,

$$s(1) = 20 - 5 = 15.$$

At $t = 1 + h$,

$$\begin{aligned} s(1+h) &= 20(1+h) - 5(1+h)^2 \\ &= 20 + 20h - 5(1 + 2h + h^2) \\ &= 15 + 10h - 5h^2. \end{aligned}$$

Simplify the average velocity

For $h \neq 0$,

$$\begin{aligned}\frac{s(1+h)-s(1)}{h} &= \frac{15+10h-5h^2-15}{h}\\ &= \frac{10h-5h^2}{h}\\ &= 10-5h.\end{aligned}$$

The cancellation is valid before taking the limit because $h \neq 0$.

Take the limit to obtain the instantaneous velocity

Now let $h \to 0$:

$$\lim_{h \to 0} (10 - 5h) = 10.$$

The instantaneous velocity at $t = 1$ is 10 meters per second.

In the next lecture, this limiting rate will receive a name: the derivative.

Closing summary

Continuity turns local control into global conclusions

- ⋆ Continuity at a means $\lim_{x \to a} f(x) = f(a)$.
- ⋆ Bolzano guarantees a zero after a continuous sign change.
- ⋆ The Intermediate Value Theorem guarantees every intermediate height.
- ⋆ The Extreme Value Theorem guarantees attained extrema on $[a, b]$.
- ⋆ Difference quotients connect secants, tangents and velocities.

Final check

Suppose $f : [-1, 2] \to \mathbb{R}$ is continuous and

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What can we conclude immediately?

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Suppose $f : [-1, 2] \to \mathbb{R}$ is continuous and

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What can we conclude immediately?

The value 0 lies between -3 and 5. Therefore, the Intermediate Value Theorem gives a point $c \in (-1, 2)$ such that

$$\boxed{f(c) = 0}.$$

It does not guarantee that this point is unique.