Limits and Continuity

Exercise 1. Calculate the following limits:

a) $\displaystyle\lim_{x\to 0} \frac{x+x^2}{|x|}$, b) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x-1}}{|x-1|}$, c) $\displaystyle\lim_{x\to 0} \frac{2x-|x|}{|3x|-2x}$.

Exercise 2. Calculate the following limits:

a) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$,

b) $\displaystyle\lim_{x\to 5} \frac{4-\sqrt{3x+1}}{x^2-7x+10}$,

c) $\displaystyle\lim_{x\to\infty} \frac{x^2+x^3}{\sin(x)+\sqrt{x}}$,

d) $\displaystyle\lim_{x\to+\infty} \frac{x^2+2e^{-x}+e^x}{3e^x+5e^{-x}+\log(\sqrt{x})}$,

e) $\displaystyle\lim_{x\to+\infty} (\sqrt{x})^{\frac{1}{\log(x^2)}}$,

f) $\displaystyle\lim_{x\to+\infty} \sqrt{x}\left(\log(\sqrt{x})-\log(\sqrt{x}+5)\right)$,

g) $\displaystyle\lim_{x\to 0} \arctan\left(\frac{7}{x}\right)-\arctan\left(\frac{-5}{x}\right)$.

Exercise 3. Let $f:\ \mathbb{R}^* \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \left(\frac{1}{x}-\frac{1}{4}\right)\left(\frac{1}{x-4}\right), & \text{if } x \neq 4 \\ a, & \text{if } x = 4 \end{cases}$$

where $a \in \mathbb{R}$. Find the value of a for which the function f is continuous on all of $\mathbb{R}$. Also study the limits of f at $+\infty$ and $-\infty$.

Exercise 4. Let $f:\ \mathbb{R}^+ \setminus \{e\} \to \mathbb{R}$ be the function defined by $f(x) = x^{\frac{1}{\log(x)-1}}$ for every $x \in \mathbb{R}^+ \setminus \{e\}$. Study the behaviour of f at 0, e, and $+\infty$.

Exercise 5. Let $f:\ (0,\pi/2) \to \mathbb{R}$ be the function defined by $f(x) = \left(\frac{1}{\tan(x)}\right)^{\sin(x)}$. Prove that f has a limit at 0 and at $\pi/2$, and calculate these limits.

Exercise 6. Let $f:\ (0,\pi/2) \to \mathbb{R}$ be the function defined by

$$f(x) = \left(1+\sin(x)\right)^{\cot(x)}.$$

Study the continuity of f and its behaviour at 0 and $\pi/2$.

Exercise 7. Prove that there exists a positive real number x such that $\log(x)+\sqrt{x}=0$.

Exercise 8. Prove that the equation $2^{-x^2} = x$ has at least one solution.

Exercise 9. Prove that the equation $x + e^x + \arctan(x) = 0$ has exactly one real root. Give an interval of length one containing this root.

Exercise 10. Determine the range of the function $f:\ \mathbb{R}^* \to \mathbb{R}$ defined by $f(x) = \arctan(\log|x|)$.

Exercise 11. Let $f:\ [0,\pi/2] \to \mathbb{R}$ be defined by $f(x) = \cos(x)$. Prove that f has a fixed point; that is, there exists $c \in [0,\pi/2]$ such that $\cos(c) = c$.