Calculus 1

Week 2

Problem 1. Determine whether the following limit exists $\lim_{x \to +\infty} \sin x$.

Solution. Consider the sequences

$$x_n = \frac{\pi}{2} + 2\pi n, \qquad y_n = \frac{3\pi}{2} + 2\pi n.$$

Both sequences tend to $+\infty$, but

$$\sin x_n = 1 \quad \text{and} \quad \sin y_n = -1.$$

The function approaches different values along two sequences tending to $+\infty$. Consequently, the limit does not exist.

Problem 2. Determine whether the following limit exists $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.

Solution. Let

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n} \quad \text{and} \quad y_n = \frac{1}{\frac{3\pi}{2} + 2\pi n}.$$

Then $x_n \to 0$ and $y_n \to 0$, whereas

$$\sin\left(\frac{1}{x_n}\right) = 1 \quad \text{and} \quad \sin\left(\frac{1}{y_n}\right) = -1.$$

Thus, $\sin(1/x)$ oscillates between different values arbitrarily close to 0 and therefore the limit does not exist.

Problem 3. Prove directly from the ε–δ definition that $\lim_{x \to 0}(x^2 + 1) = 1$.

Solution. Let $\varepsilon > 0$ and choose $\delta = \sqrt{\varepsilon}$. If $0 < |x| < \delta$, then

$$\left|(x^2 + 1) - 1\right| = x^2 < \delta^2 = \varepsilon.$$

Therefore, the required limit is 1.

Before solving the next problem, let us give the definition of infinite limit formally.

Definition. Let $f : A \to \mathbb{R}$, and let a be a point that can be approached by points of A different from a. We say that

$$\lim_{x \to a} f(x) = +\infty$$

if, for every $M > 0$, there exists $\delta > 0$ such that

$$x \in A, \quad 0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > M.$$

Problem 4. Prove from the definition of an infinite limit that $\lim_{x\to 0} \frac{1}{x^2} = +\infty$.

Solution. Let $M > 0$ and choose
$$\delta = \frac{1}{\sqrt{M}}.$$
If $0 < |x| < \delta$, then
$$x^2 < \delta^2 = \frac{1}{M},$$
and consequently
$$\frac{1}{x^2} > M.$$
This proves that the limit is $+\infty$.

Problem 5. Prove directly that $\lim_{x\to+\infty} (x+1) = +\infty$.

Solution. Let $M > 0$ and choose $N = \max\{0, M-1\}$. If $x > N$, then $x > M - 1$ and hence $x + 1 > M$. Therefore, $x + 1 \to +\infty$ as $x \to +\infty$.

Problem 6. Prove that, for every $a \in \mathbb{R}$, $\lim_{x\to a} x = a$.

Solution. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If
$$0 < |x - a| < \delta,$$
then
$$|x - a| < \varepsilon.$$
This is precisely the ε–δ condition for the stated limit.

Problem 7. Let $k \in \mathbb{R}$. Prove that, for every $a \in \mathbb{R}$, $\lim_{x\to a} k = k$.

Solution. Let $\varepsilon > 0$. We may choose any $\delta > 0$, for example $\delta = 1$. Then
$$|k - k| = 0 < \varepsilon$$
whenever $0 < |x - a| < \delta$. Thus, the limit of a constant function is the same constant.

Problem 8. Let $f(x) = \sqrt{x}$, whose natural domain is $[0, +\infty)$. Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$.

Solution. Let $\varepsilon > 0$ and choose $\delta = \varepsilon^2$. If $0 < x < \delta$, then
$$0 < \sqrt{x} < \sqrt{\delta} = \varepsilon.$$
Therefore, the right-hand limit is 0.

Problem 9. Prove directly that $\lim_{x\to 0^+} \ln x = -\infty$.

Solution. Let $M > 0$ and choose $\delta = e^{-M}$. If $0 < x < \delta$, then, since the logarithm is increasing,

$$\ln x < \ln \delta = \ln(e^{-M}) = -M.$$

Consequently, $\ln x \to -\infty$ as $x \to 0^+$.

Problem 10. Study the one-sided limits of $f(x) = 1/x$ at 0, and determine whether the two-sided limit exists.

Solution. Let $M > 0$ and choose $\delta = 1/M$. If $0 < x < \delta$, then

$$\frac{1}{x} > M,$$

so

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

If $-\delta < x < 0$, then

$$\frac{1}{x} < -M,$$

and hence

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Since the one-sided limits are different, the two-sided limit does not exist.

Problem 11. Prove directly from the ε–δ definition that

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{x - 1} = 3.$$

Solution. For $x \neq 1$,

$$\frac{2x^2 - x - 1}{x - 1} = \frac{(2x+1)(x-1)}{x-1} = 2x + 1.$$

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/2$. If $0 < |x - 1| < \delta$, then

$$\left| \frac{2x^2 - x - 1}{x - 1} - 3 \right| = |2x + 1 - 3| = 2\,|x - 1| < 2\delta = \varepsilon.$$

Thus, the required limit is 3.

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$ and $g : \mathbb{R} \setminus \{2\} \to \mathbb{R}$ defined by $g(y) = \frac{y^2-4}{y-2}$. We have

$$\lim_{x \to 2} f(x) = 2 \qquad \text{and} \qquad \lim_{y \to 2} g(y) = 4.$$

Can the composition theorem be used to calculate $\lim_{x \to 2} (g \circ f)(x)$?

Solution. For $y \neq 2$,

$$g(y) = \frac{(y-2)(y+2)}{y-2} = y + 2,$$

so $\lim_{y \to 2} g(y) = 4$. However, $f(x) = 2$ for every $x \in \mathbb{R}$, while $2 \notin \mathrm{Dom}(g)$. Therefore,

$$(g \circ f)(x) = g(2)$$

is not defined for any x. The composition theorem cannot be applied: g is not defined at the limiting value 2, and $f(x)$ does not avoid that value near $x = 2$.

Problem 13. Calculate the following limits:

(a) $\lim_{x \to +\infty} (3x^3 - 2x)$

(b) $\lim_{x \to -\infty} (3x^3 - 2x)$

(c) $\lim_{x \to +\infty} \dfrac{x^2 + 7}{x - 2}$

(d) $\lim_{x \to +\infty} \dfrac{2x + 7}{x - 1}$

(e) $\lim_{x \to +\infty} \dfrac{3x - 1}{x^2 + 1}$

Solution.

(a) We factor out the dominant power:

$$3x^3 - 2x = x^3 \left(3 - \frac{2}{x^2}\right).$$

The second factor tends to 3, whereas $x^3 \to +\infty$. Therefore, the limit is $+\infty$.

(b) The same factorization gives

$$x^3 \to -\infty, \qquad 3 - \frac{2}{x^2} \to 3 > 0.$$

Hence, the limit is $-\infty$.

(c) We write

$$\frac{x^2 + 7}{x - 2} = x \frac{1 + 7/x^2}{1 - 2/x}.$$

The quotient tends to 1, while $x \to +\infty$. Thus, the limit is $+\infty$.

(d) Dividing the numerator and denominator by x gives

$$\frac{2x + 7}{x - 1} = \frac{2 + 7/x}{1 - 1/x} \longrightarrow 2.$$

(e) Dividing the numerator and denominator by x^2 gives

$$\frac{3x - 1}{x^2 + 1} = \frac{3/x - 1/x^2}{1 + 1/x^2} \longrightarrow 0.$$

Problem 14. Calculate $\lim_{x \to +\infty} \dfrac{\cos x}{x}$.

Solution. For $x > 0$, we have

$$\left|\frac{\cos x}{x}\right| \leqslant \frac{1}{x}.$$

Since $1/x \to 0$ as $x \to +\infty$, the squeeze theorem gives

$$\lim_{x\to+\infty} \frac{\cos x}{x} = 0.$$

Problem 15. Calculate $\lim_{x\to+\infty}(x + \sin x)$.

Solution. Since $\sin x \geqslant -1$, we have

$$x + \sin x \geqslant x - 1.$$

Moreover, $x - 1 \to +\infty$. By comparison,

$$\lim_{x\to+\infty}(x + \sin x) = +\infty.$$

Problem 16. Calculate $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$.

Solution. Since

$$\left|\sin\left(\frac{1}{x}\right)\right| \leqslant 1,$$

we have

$$\left|x \sin\left(\frac{1}{x}\right)\right| \leqslant |x|.$$

Because $|x| \to 0$ as $x \to 0$, the squeeze theorem yields

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Problem 17. Prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Solution. For $0 < x < \pi/2$, the geometric inequalities

$$\sin x < x < \tan x$$

hold. Dividing by $\sin x > 0$ gives

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Taking reciprocals, we obtain

$$\cos x < \frac{\sin x}{x} < 1.$$

Both outer expressions tend to 1 as $x \to 0^+$. Hence,

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1.$$

The function $\sin x/x$ is even, so the left-hand limit is also 1. Therefore,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Problem 18. Calculate $\lim\limits_{x\to+\infty} x\sin\left(\frac{1}{x}\right)$.

Solution. We write

$$x\sin\left(\frac{1}{x}\right) = \frac{\sin(1/x)}{1/x}.$$

Putting $u = 1/x$, we have $u \to 0^+$. Thus,

$$\lim_{x\to+\infty} x\sin\left(\frac{1}{x}\right) = \lim_{u\to 0^+} \frac{\sin u}{u} = 1.$$

Problem 19. Calculate $\lim\limits_{x\to 0} \frac{\sin(x^3)}{x}$.

Solution. For $x \neq 0$,

$$\frac{\sin(x^3)}{x} = x^2\frac{\sin(x^3)}{x^3}.$$

As $x \to 0$,

$$x^2 \to 0 \qquad \text{and} \qquad \frac{\sin(x^3)}{x^3} \to 1.$$

Consequently,

$$\lim_{x\to 0} \frac{\sin(x^3)}{x} = 0.$$

Problem 20. Let $g : [0, +\infty) \to \mathbb{R}$ be defined by $g(x) = \sqrt{x}$. Verify directly that g is continuous from the right at 0.

Solution. We have $g(0) = 0$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon^2$. If $x \in [0, +\infty)$ and $0 \leqslant x < \delta$, then

$$|g(x) - g(0)| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

Therefore,

$$\lim_{x\to 0^+} g(x) = g(0) = 0,$$

and g is continuous from the right at 0.

Problem 21. Show that the polynomial

$$p(x) = 2x^3 - 5x^2 - 10x + 5$$

has at least one zero in $[-1, 2]$. Find a smaller interval containing a zero.

Solution. The function p is a polynomial, so it is continuous on $[-1, 2]$. Moreover,

$$p(-1) = 8 > 0, \qquad p(2) = -19 < 0.$$

By Bolzano's theorem, there is a point $c \in (-1, 2)$ such that $p(c) = 0$.

We can improve the interval because

$$p(0) = 5 > 0, \qquad p\left(\frac{1}{2}\right) = -1 < 0.$$

Applying Bolzano's theorem once more shows that, in fact, a zero lies in $(0, 1/2)$.

Problem 22. Calculate $\lim_{x\to 0^-} \left(e^{-1/x} + \frac{x+4}{x^2+1}\right)$.

Solution. As $x \to 0^-$, we have $-1/x \to +\infty$, and hence

$$e^{-1/x} \to +\infty.$$

The rational term has the finite limit

$$\frac{x+4}{x^2+1} \longrightarrow 4.$$

Therefore,

$$\lim_{x\to 0^-} \left(e^{-1/x} + \frac{x+4}{x^2+1}\right) = +\infty.$$

Problem 23. Calculate $\lim_{x\to 1^+} e^{x+\ln(x-1)}$.

Solution. For $x > 1$,

$$e^{x+\ln(x-1)} = e^x e^{\ln(x-1)} = e^x(x-1).$$

As $x \to 1^+$, the first factor tends to e and the second tends to 0. Hence,

$$\lim_{x\to 1^+} e^{x+\ln(x-1)} = 0.$$

Problem 24. Calculate $\lim_{x\to\pi} \arctan\left(\frac{1}{1+\cos x}\right)$.

Solution. Using the identity

$$1 + \cos x = 2\cos^2\left(\frac{x}{2}\right),$$

we see that $1 + \cos x \to 0^+$ as $x \to \pi$. Therefore,

$$\frac{1}{1+\cos x} \to +\infty.$$

Since $\arctan t \to \pi/2$ as $t \to +\infty$, it follows that

$$\lim_{x\to\pi} \arctan\left(\frac{1}{1+\cos x}\right) = \frac{\pi}{2}.$$

Problem 25. Determine whether the following limit exists:

$$\lim_{x\to 0} \arctan\left(\frac{1}{\sin x}\right).$$

Solution. If $x \to 0^+$, then $\sin x \to 0^+$, so

$$\frac{1}{\sin x} \to +\infty \qquad \text{and} \qquad \arctan\left(\frac{1}{\sin x}\right) \to \frac{\pi}{2}.$$

If $x \to 0^-$, then $\sin x \to 0^-$, so

$$\frac{1}{\sin x} \to -\infty \qquad \text{and} \qquad \arctan\left(\frac{1}{\sin x}\right) \to -\frac{\pi}{2}.$$

The one-sided limits are different. Consequently, the two-sided limit does not exist.

Problem 26. Calculate $\lim\limits_{x\to-\infty} \dfrac{e^x}{x-1}$.

Solution. Put $t = -x$. Then $t \to +\infty$ and

$$\frac{e^x}{x-1} = -\frac{e^{-t}}{t+1}.$$

For $t > 0$,

$$0 \leqslant \frac{e^{-t}}{t+1} \leqslant e^{-t}.$$

The right-hand side tends to 0, so the squeeze theorem gives

$$\lim_{x\to-\infty} \frac{e^x}{x-1} = 0.$$

Problem 27. Calculate $\lim\limits_{x\to+\infty} \ln\left(\dfrac{1+3x^2}{x^2+1}\right)$.

Solution. Dividing the numerator and denominator inside the logarithm by x^2, we obtain

$$\frac{1+3x^2}{x^2+1} = \frac{3+1/x^2}{1+1/x^2} \longrightarrow 3.$$

The logarithm is continuous at 3. Therefore,

$$\lim_{x\to+\infty} \ln\left(\frac{1+3x^2}{x^2+1}\right) = \ln 3.$$

Problem 28. Calculate $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

Solution. We rationalize the numerator:

$$\begin{aligned}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} \\ &= \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \frac{2}{\sqrt{1+x}+\sqrt{1-x}}.\end{aligned}$$

Thus,

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\frac{2}{1+1}=1.$$

Problem 29. Calculate the following limits:

(a) $\displaystyle\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}}$;

(b) $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}}$.

Solution. The radicand must be positive because it appears under a square root in the denominator. Thus,

$$x^2+x=x(x+1)>0,$$

and the natural domain is

$$(-\infty,-1)\cup(0,+\infty).$$

For every x in this domain,

$$\sqrt{x^2+x}=\sqrt{x^2\left(1+\frac{1}{x}\right)}=|x|\sqrt{1+\frac{1}{x}}.$$

(a) When $x>0$, we have $|x|=x$. Therefore,

$$\frac{x}{\sqrt{x^2+x}}=\frac{1}{\sqrt{1+1/x}}\longrightarrow 1.$$

(b) When $x<0$, we have $|x|=-x$. Hence,

$$\frac{x}{\sqrt{x^2+x}}=-\frac{1}{\sqrt{1+1/x}}\longrightarrow -1.$$

The different signs come from the identity $\sqrt{x^2}=|x|$.

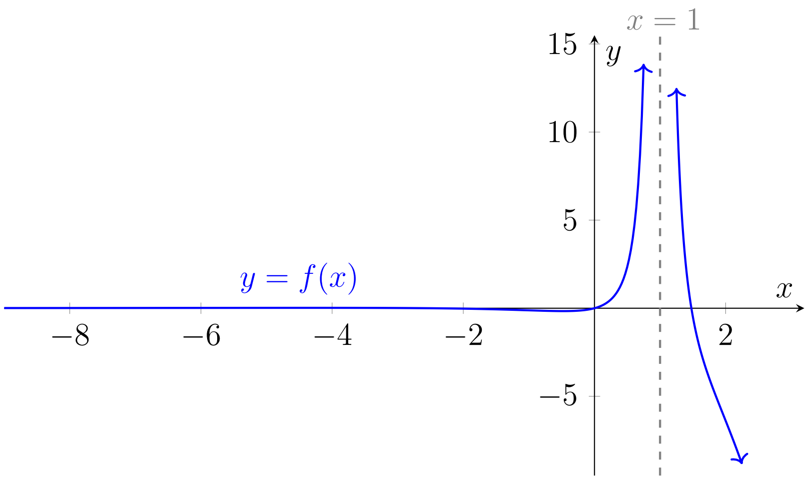
Problem 30. Draw the graph of a function f satisfying

$$\lim_{x\to-\infty}f(x)=0,\qquad \lim_{x\to 1}f(x)=+\infty,\qquad \lim_{x\to+\infty}f(x)=-\infty.$$

Solution. One possible function is

$$f(x)=\frac{1}{(x-1)^2}-e^x,\qquad x\neq 1.$$

Indeed, as $x\to-\infty$, both $1/(x-1)^2$ and e^x tend to 0, so $f(x)\to 0$. As $x\to 1$, the first term tends to $+\infty$, while e^x remains finite. Finally, as $x\to+\infty$, the exponential term dominates and $f(x)\to-\infty$.



Problem 31. Let
$$g(x) = e^{-1/(1+x)}.$$
Find the natural domain of g and calculate its limits at the boundary points of the intervals in its domain.

Solution. The denominator $1 + x$ cannot vanish. Therefore,
$$\mathrm{Dom}(g) = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, +\infty).$$
As $x \to -\infty$, the exponent $-1/(1 + x)$ tends to 0^+, whereas as $x \to +\infty$ it tends to 0^-. Hence,
$$\lim_{x\to-\infty} g(x) = 1, \qquad \lim_{x\to+\infty} g(x) = 1.$$
At the finite boundary point,
$$-\frac{1}{1+x} \to +\infty \quad \text{as } x \to -1^-,$$
and
$$-\frac{1}{1+x} \to -\infty \quad \text{as } x \to -1^+.$$
Consequently,
$$\lim_{x\to-1^-} g(x) = +\infty, \qquad \lim_{x\to-1^+} g(x) = 0.$$

Problem 32. Consider the real-valued function
$$h(x) = \sqrt[x-1]{\arctan x} = (\arctan x)^{1/(x-1)}.$$
Find its natural domain and calculate its limits at the boundary points of the intervals in its domain.

Solution. For a variable real exponent, we use the convention
$$h(x) = \exp\left(\frac{\ln(\arctan x)}{x-1}\right).$$

Thus, we need $\arctan x > 0$, which is equivalent to $x > 0$, and we must also have $x \neq 1$. Hence,

$$\mathrm{Dom}(h) = (0,1) \cup (1, +\infty).$$

As $x \to 0^+$, we have $\ln(\arctan x) \to -\infty$ and $x - 1 \to -1$. Therefore,

$$\frac{\ln(\arctan x)}{x-1} \to +\infty, \qquad \lim_{x\to 0^+} h(x) = +\infty.$$

As $x \to 1$, the numerator tends to

$$\ln\left(\frac{\pi}{4}\right) < 0.$$

Dividing by $x - 1$ gives $+\infty$ from the left and $-\infty$ from the right. Thus,

$$\lim_{x\to 1^-} h(x) = +\infty, \qquad \lim_{x\to 1^+} h(x) = 0.$$

Finally, as $x \to +\infty$,

$$\ln(\arctan x) \to \ln\left(\frac{\pi}{2}\right) \qquad \text{and} \qquad x - 1 \to +\infty.$$

The exponent tends to 0, so

$$\lim_{x\to +\infty} h(x) = e^0 = 1.$$