Calculus 1

Practice 2: Limits and Continuity

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Seven problems connect limits with continuity

1. Algebraic indeterminate forms
2. Standard limits
3. One-sided limits
4. Limits at infinity
5. Continuity of a piecewise function
6. Classification of discontinuities
7. Fixed points and instantaneous velocity

Problem 1: Algebraic limits

Direct substitution diagnoses the obstacle

For a quotient $p(x)/q(x)$, first try direct substitution.

If substitution gives a real number, the limit is usually immediate.
If it gives

$$\frac{0}{0},$$

we have an **indeterminate form**, not an answer.

The expression must be rewritten before substitution is tried again.

Problem 1 contains three different algebraic repairs

Calculate

$$
\begin{aligned}
&\text{(a)} \quad \lim_{x\to 3} \frac{x^2-9}{x-3}, \\
&\text{(b)} \quad \lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}, \\
&\text{(c)} \quad \lim_{x\to 2} \frac{1/x-1/2}{x-2}.
\end{aligned}
$$

The tools will be factorization, rationalization, and combining fractions.

Problem 1(a): substitution gives 0/0

Consider

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}.$$

Direct substitution gives

$$\frac{3^2 - 9}{3 - 3} = \frac{0}{0}.$$

The numerator is a difference of squares:

$$x^2 - 9 = (x - 3)(x + 3).$$

Problem 1(a): factor and cancel before substituting

For $x \neq 3$,

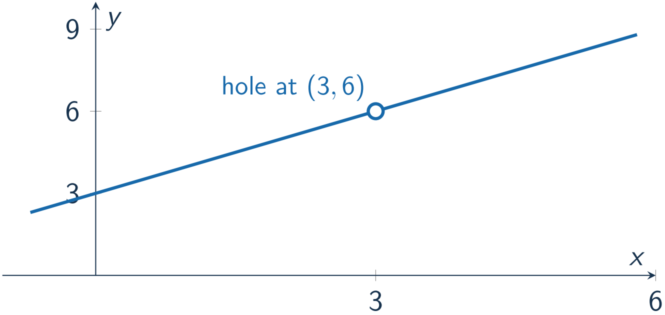
$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3}$$
$$= x+3.$$

Therefore,

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

6

The cancelled factor removes a hole only for the limit



The original quotient is undefined at $x = 3$, but its nearby values approach 6.

Problem 1(b): the conjugate removes the square root difference

Consider

$$\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4}.$$

Substitution again gives $0/0$.

Multiply by the conjugate divided by itself:

$$\frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3}.$$

This uses $(a - b)(a + b) = a^2 - b^2$.

Problem 1(b): rationalize and cancel

For $x \neq 4$,

$$\begin{aligned}
\frac{\sqrt{x+5}-3}{x-4} &= \frac{(\sqrt{x+5}-3)(\sqrt{x+5}+3)}{(x-4)(\sqrt{x+5}+3)} \\
&= \frac{x+5-9}{(x-4)(\sqrt{x+5}+3)} \\
&= \frac{1}{\sqrt{x+5}+3}.
\end{aligned}$$

Problem 1(b): substitute into the simplified expression

Now the denominator is nonzero at $x = 4$, so

$$\lim_{x\to4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

$1/6$

Problem 1(c): combine the fractions in the numerator

Consider

$$\lim_{x \to 2} \frac{1/x - 1/2}{x - 2}.$$

Use the common denominator $2x$:

$$\frac{1}{x} - \frac{1}{2} = \frac{2 - x}{2x} = -\frac{x - 2}{2x}.$$

The factor $x - 2$ is now visible.

Problem 1(c): cancel and evaluate

For $x \neq 2$,

$$\frac{1/x - 1/2}{x-2} = \frac{-(x-2)/(2x)}{x-2}$$
$$= -\frac{1}{2x}.$$

Therefore,

$$\lim_{x\to 2} \frac{1/x - 1/2}{x-2} = -\frac{1}{2\cdot 2} = -\frac{1}{4}.$$

$$-1/4$$

Problem 2: Standard limits

Four standard limits provide the basic patterns

As $u \to 0$,

$$\frac{\sin u}{u} \longrightarrow 1, \qquad \frac{1-\cos u}{u^2} \longrightarrow \frac{1}{2},$$
$$\frac{e^u - 1}{u} \longrightarrow 1, \qquad \frac{\ln(1+u)}{u} \longrightarrow 1.$$

Also,

$$(1+u)^{1/u} \longrightarrow e.$$

The main task is to rewrite each expression so that one of these patterns appears.

Problem 2(a): create the quotient $\sin u/u$

Consider

$$\lim_{x\to 0} \frac{\sin(4x)}{x}.$$

Multiply and divide by 4:

$$\frac{\sin(4x)}{x} = 4\frac{\sin(4x)}{4x}.$$

Since $4x \to 0$,

$$\frac{\sin(4x)}{4x} \longrightarrow 1.$$

$$4$$

Problem 2(b): use the substitution $u = 3x$

Let $u = 3x$. Then $u \to 0$ and $x = u/3$, so

$$\begin{aligned}\frac{1-\cos(3x)}{x^2} &= \frac{1-\cos u}{(u/3)^2} \\ &= 9\frac{1-\cos u}{u^2}.\end{aligned}$$

Therefore,

$$\lim_{x\to 0}\frac{1-\cos(3x)}{x^2} = 9\cdot\frac{1}{2} = \frac{9}{2}.$$

$$9/2$$

Problem 2(c): separate two standard limits

Rewrite

$$\frac{e^{5x}-1}{\ln(1+2x)} = \frac{e^{5x}-1}{5x} \cdot \frac{2x}{\ln(1+2x)} \cdot \frac{5}{2}.$$

As $x \to 0$,

$$\frac{e^{5x}-1}{5x} \to 1, \qquad \frac{2x}{\ln(1+2x)} \to 1.$$

Hence the product tends to $1 \cdot 1 \cdot 5/2$.

$$5/2$$

Problem 2(d): recognize the form 1^∞

Consider

$$\lim_{x \to 0} (1 + 3x)^{2/x}.$$

The base tends to 1, while the exponent is unbounded.

Rewrite the exponent so that the standard pattern appears:

$$\frac{2}{x} = 6 \cdot \frac{1}{3x}.$$

Thus,

$$(1 + 3x)^{2/x} = \left((1 + 3x)^{1/(3x)}\right)^6.$$

Problem 2(d): apply the exponential standard limit

Since $3x \to 0$,

$$(1+3x)^{1/(3x)} \longrightarrow e.$$

Taking the sixth power gives

$$\lim_{x \to 0} (1+3x)^{2/x} = e^6.$$

$$e^6$$

The expression is defined near 0 because $1+3x>0$ whenever x is sufficiently close to 0.

Problem 3: One-sided limits

A two-sided limit needs agreement from both sides

If both sides of a belong to the domain near a, then

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L.$$

Different one-sided limits imply that the two-sided limit does not exist.

Problem 3(a): determine the formula on each side of -1

Let

$$f(x) = \frac{x+1}{|x+1|}.$$

If $x < -1$, then $x + 1 < 0$ and

$$|x+1| = -(x+1), \qquad f(x) = -1.$$

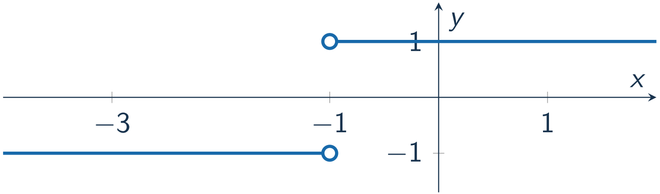
If $x > -1$, then $x + 1 > 0$ and

$$|x+1| = x+1, \qquad f(x) = 1.$$

Problem 3(a): the one-sided limits disagree

Therefore,

$$\lim_{x\to -1^-} f(x) = -1, \qquad \lim_{x\to -1^+} f(x) = 1.$$



The two-sided limit does not exist.

Problem 3(b): the sign of the denominator changes at 2

Let

$$g(x) = \frac{1}{x-2}.$$

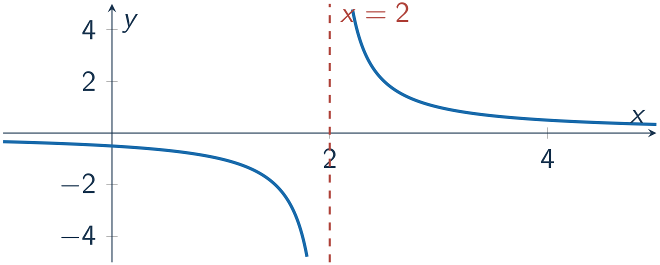
As $x \to 2^-$, the denominator is negative and approaches 0:

$$\frac{1}{x-2} \longrightarrow -\infty.$$

As $x \to 2^+$, the denominator is positive and approaches 0:

$$\frac{1}{x-2} \longrightarrow +\infty.$$

Problem 3(b): the vertical asymptote separates the two sides



$\lim_{x\to 2^-} g(x) = -\infty$, $\lim_{x\to 2^+} g(x) = +\infty$; no two-sided limit.

Problem 3(c): $|x|/x$ is the sign of x

For

$$h(x) = \frac{|x|}{x},$$

we have

$$h(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Hence,

$$\lim_{x \to 0^-} h(x) = -1, \qquad \lim_{x \to 0^+} h(x) = 1.$$

The two-sided limit does not exist.

Problems 3(a) and 3(c) have the same shape

The two expressions differ only by a horizontal translation:

$$\frac{x+1}{|x+1|} \qquad \text{and} \qquad \frac{|x|}{x}.$$

Both take the value -1 on the left and 1 on the right.
The jump occurs at -1 in the first case and at 0 in the second.

A horizontal shift changes the location of the jump, not its type.

Problem 4: Limits at infinity

Choose the method from the dominant structure

For limits as $x \to +\infty$:

- Rational functions: divide by the highest power of x.
- Differences involving square roots: rationalize first.
- Expressions containing $1/x$: substitute $u = 1/x$.

Terms containing $1/x^p$ tend to zero when $p > 0$.

Problem 4(a): divide by the highest power

Consider

$$\lim_{x\to+\infty} \frac{5x^3 - 2x + 1}{2x^3 + x^2}.$$

Divide the numerator and denominator by x^3:

$$\frac{5 - 2/x^2 + 1/x^3}{2 + 1/x}.$$

As $x \to +\infty$, every term containing $1/x$ tends to 0. Therefore, the quotient tends to $(5 - 0 + 0)/(2 + 0)$.

$$5/2$$

Problem 4(b): rationalize the difference

Consider
$$\sqrt{x^2+3x}-x.$$
This has the indeterminate form $\infty-\infty$.
Multiply by the conjugate:
$$\begin{aligned}\sqrt{x^2+3x}-x &= \frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{\sqrt{x^2+3x}+x}\\ &= \frac{3x}{\sqrt{x^2+3x}+x}.\end{aligned}$$

Problem 4(b): factor x from the denominator

Because $x \to +\infty$, we have $x > 0$ eventually. Thus

$$\sqrt{x^2 + 3x} = x\sqrt{1 + \frac{3}{x}}.$$

Hence,

$$\frac{3x}{\sqrt{x^2 + 3x} + x} = \frac{3}{\sqrt{1 + 3/x} + 1}.$$

Now let $x \to +\infty$:

$$\frac{3}{\sqrt{1 + 0} + 1} = \frac{3}{2}.$$

$3/2$

Problem 4(c): turn infinity into a limit at zero

Consider

$$\lim_{x \to +\infty} x \sin(1/x).$$

Set

$$u = \frac{1}{x}.$$

Then $u \to 0^+$ and $x = 1/u$. Therefore, the standard trigonometric limit gives

$$x \sin(1/x) = \frac{\sin u}{u} \longrightarrow 1.$$

$$1$$

Problem 5: Piecewise continuity

Only the junction points require attention

Find $a, b \in \mathbb{R}$ so that

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x < 1, \\ ax + b, & 1 \leq x < 2, \\ 5, & x \geq 2 \end{cases}$$

is continuous on $\mathbb{R}$.

Each formula is continuous inside its interval. Only $x = 1$ and $x = 2$ can cause problems.

Simplify the branch to the left of 1

For $x < 1$, we automatically have $x \neq 1$, so

$$\begin{aligned} \frac{x^2-1}{x-1} &= \frac{(x-1)(x+1)}{x-1} \\ &= x+1. \end{aligned}$$

Therefore,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+1) = 2.$$

Continuity at $x = 1$ gives the first equation

The middle formula supplies both the value and the right-hand limit:

$$f(1) = a + b, \qquad \lim_{x \to 1^+} f(x) = a + b.$$

Continuity requires agreement with the left-hand limit 2:

$$a + b = 2.$$

First junction condition: $a + b = 2$

Continuity at $x = 2$ gives the second equation

From the middle branch,

$$\lim_{x \to 2^-} f(x) = 2a + b.$$

From the final branch,

$$f(2) = 5, \qquad \lim_{x \to 2^+} f(x) = 5.$$

Therefore, continuity at 2 requires

$$2a + b = 5.$$

Second junction condition: $2a + b = 5$

Solve the two junction equations

We have the linear system

$$\begin{cases} a + b = 2, \\ 2a + b = 5. \end{cases}$$

Subtract the first equation from the second:

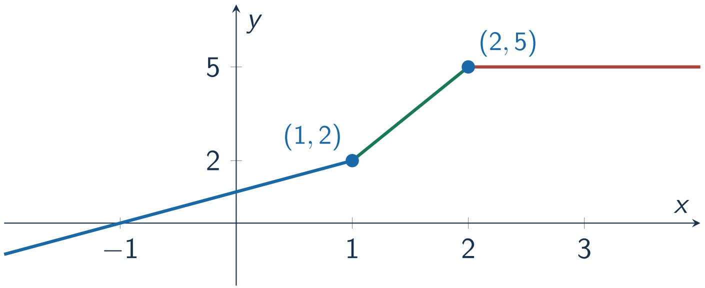
$$a = 3.$$

Substitute into $a + b = 2$:

$$3 + b = 2 \implies b = -1.$$

$$a = 3 \text{ and } b = -1$$

The three pieces now meet without gaps



At both junctions, the left limit, right limit and function value agree.

Problem 6: Discontinuities

The behavior of the limit determines the type

Type	Behavior near a	Repair one value?
Removable	finite limit exists	yes
Jump	finite one-sided limits differ	no
Infinite	values become unbounded	no

Changing $f(a)$ affects only one point; it does not change nearby values or one-sided limits.

Problem 6(a): simplify away from the exceptional point

For $x \neq 2$,

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x-2} = x + 2.$$

Therefore,

$$\lim_{x \to 2} f(x) = 4.$$

But the given value is

$$f(2) = 7.$$

Problem 6(a): changing one value fills the hole

Since

$$\lim_{x \to 2} f(x) = 4 \neq 7 = f(2),$$

the discontinuity is removable.

Keep the same formula away from 2 and define the repaired value by $f(2) = 4$. Then

$$\lim_{x \to 2} f(x) = f(2) = 4,$$

so the repaired function is continuous at 2.

Removable; replace $f(2) = 7$ by $f(2) = 4$.

Problem 6(b): the one-sided limits form a jump

Let

$$g(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

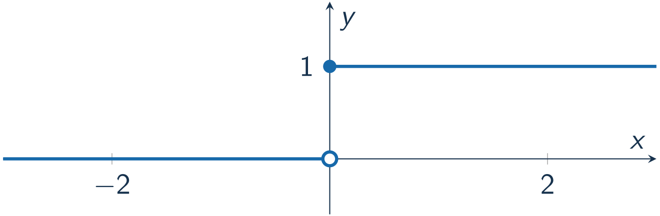
Then

$$\lim_{x \to 0^-} g(x) = 0, \qquad \lim_{x \to 0^+} g(x) = 1.$$

The two one-sided limits are finite but different.

Jump discontinuity at 0.

Changing $g(0)$ cannot close a jump



Changing the value at 0 moves only the filled point. The two horizontal pieces still approach different heights.

Problem 6(c): the function becomes unbounded near 1

For

$$h(x) = \frac{1}{(x-1)^2},$$

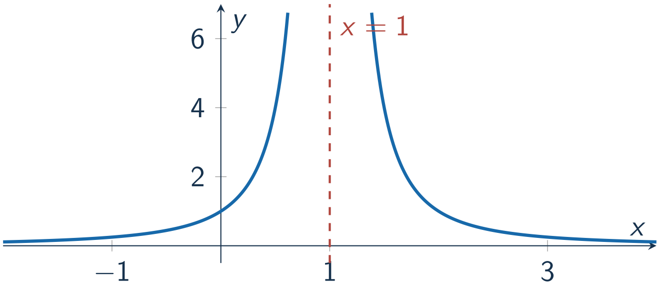
the denominator is positive on both sides of 1 and tends to zero. Hence,

$$\lim_{x \to 1^-} h(x) = +\infty, \qquad \lim_{x \to 1^+} h(x) = +\infty.$$

This is an infinite discontinuity with vertical asymptote $x = 1$.

No finite choice of $h(1)$ can repair it.

One point cannot replace an infinite vertical branch



The nearby values are unbounded, independently of any value assigned at $x = 1$.

Only the removable discontinuity can be repaired locally

Case	Classification	Change one value?
(a)	removable at 2	yes: set $f(2) = 4$
(b)	jump at 0	no
(c)	infinite at 1	no

A one-point repair is possible exactly when a finite two-sided limit exists.

Problem 7: Two applications

Problem 7(a): convert a fixed point into a zero

Let

$$f : [0, 1] \longrightarrow [0, 1]$$

be continuous. A fixed point is a number c satisfying

$$f(c) = c.$$

Define

$$F(x) = f(x) - x.$$

Then F is continuous, and

$$F(c) = 0 \quad \Longleftrightarrow \quad f(c) = c.$$

The endpoint values have opposite weak signs

Because $f(x) \in [0, 1]$ for every $x \in [0, 1]$,

$$F(0) = f(0) \geq 0,$$

and

$$F(1) = f(1) - 1 \leq 0.$$

Thus F is non-negative at the left endpoint and non-positive at the right endpoint.

Bolzano's theorem gives the fixed point

There are two possibilities:

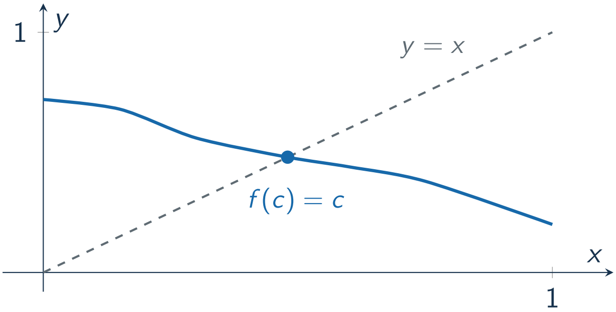
- If $F(0) = 0$ or $F(1) = 0$, an endpoint is already a fixed point.
- Otherwise, $F(0) > 0 > F(1)$, so Bolzano's theorem gives $c \in (0,1)$ with $F(c) = 0$.

In either case,

$$F(c) = 0 \quad \Longrightarrow \quad f(c) = c.$$

Every continuous $f : [0,1] \to [0,1]$ has a fixed point.

Geometrically, the graph must meet the diagonal



The precise crossing point is not known, but continuity guarantees that one exists.

Problem 7(b): average velocity is a difference quotient

Let

$$s(t) = t^2 - 3t.$$

The average velocity from $t = 1$ to $t = 1 + h$ is

$$\frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{s(1+h) - s(1)}{h}, \qquad h \neq 0.$$

We first calculate the two position values separately.

Calculate the positions at 1 and $1 + h$

At $t = 1$,

$$s(1) = 1^2 - 3(1) = -2.$$

At $t = 1 + h$,

$$\begin{aligned} s(1+h) &= (1+h)^2 - 3(1+h) \\ &= 1 + 2h + h^2 - 3 - 3h \\ &= -2 - h + h^2. \end{aligned}$$

Simplify the average velocity

Substitute the position values:

$$\begin{aligned}\frac{s(1+h)-s(1)}{h} &= \frac{(-2-h+h^2)-(-2)}{h}\\ &= \frac{-h+h^2}{h}\\ &= -1+h, \qquad h \neq 0.\end{aligned}$$

Average velocity from 1 to $1+h$: $-1+h$

Let the time interval shrink to obtain instantaneous velocity

The instantaneous velocity at $t = 1$ is

$$\lim_{h \to 0} \frac{s(1+h) - s(1)}{h}.$$

Using the simplified quotient,

$$\lim_{h \to 0} (-1 + h) = -1.$$

Instantaneous velocity at $t = 1$: -1

The negative sign means that the position is decreasing at that instant.

Closing check

The method is more important than the final number

- A form such as $0/0$ tells us to simplify before evaluating.
- Standard limits must be exposed by scaling or substitution.
- A two-sided limit requires equal one-sided limits.
- Continuity at a junction produces equations for the unknown parameters.
- A one-point repair works only for removable discontinuities.
- Existence theorems and difference quotients turn limits into applications.

Final check

Classify the discontinuity of

$$q(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 4, & x = 0. \end{cases}$$

Can it be repaired by changing only $q(0)$?

Final check: compare the limit with the value

The standard limit gives

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

But

$$q(0) = 4.$$

Thus the finite two-sided limit exists but differs from the function value.

The discontinuity is removable; set $q(0) = 1$.